Quantitative Methods for Economics

Tutorial 3

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TUTORIAL 3

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ECO3021S

PART 1

1. Let the national income model be:

$$\begin{aligned} Y &= C + I + G \\ C &= a + b(Y - T) \qquad a > 0, 0 < b < 1 \\ T &= d + tY \qquad d > 0, 0 < t < 1 \end{aligned}$$

   where Y is national income, C is (planned) consumption expenditure, I is investment expenditure, G is government expenditure and T is taxes.

   (a) Identify which variables are endogenous, and which are exogenous.

   (b) Give the economic meaning of the parameters a, b, d and t.

   (c) Use Cramer's Rule to solve for the equilibrium national income, consumption and taxation.

2. Determine the definiteness of the symmetric matrix

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

3. For each of the following find $\dfrac{dy}{dx}$:

   (a) $y = \dfrac{x^3 + 2}{x - 1}$

   (b) $y = \dfrac{x^2 \left(1 - x^3\right)^2}{\sqrt[3]{4 + x^2}}$

   (c) $y = e^{4x^2 + 3x - 1}$

4. Find f_x and f_y:

   $f(x, y) = x^3 + 2xy - y^3$

5. The demand for a firm's output is given by:

$$Q = 50 - 0.25P - 0.2M + 0.10P_s$$

where P is the price of the firm's good;

M is the average weekly income of consumers;

P_s is the price of the substitute good.

(a) Demonstrate using derivatives what will happen to the demand for the firm's product if the firm raises the price for its good, or if the price of the substitute good rises.

(b) Demonstrate, using derivatives, whether this is a normal or an inferior good. Motivate your answer.

(c) If we assume that $P = 12$, $M = 100$, and $P_s = 30$, what is the price elasticity of demand for the firm's product?

6. Individual A has a utility function of the form $u(c) = 4\alpha\gamma\sqrt{c}$ while individual B's utility is given by $U(C) = 5\beta\ln(C)$, where c, C represent A and B's consumption respectively. Show that the preferences of these two individuals' exhibit *local non-satiation* and *diminishing marginal utility* of consumption.

PART 2

7. Determine the definiteness of the symmetric matrices

(a) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

8. For each of the following find $\frac{dy}{dx}$:

(a) $y = (x + 7)(2x + 3)$

(b) $y = (x^2 - 2x + 1)^{15}$

(c) $y = \ln 3x + \ln(5 + x^2)$

9. Find f_x and f_y:

$z = f(x, y) = \dfrac{3x - 2y}{x^2 - y}$

10. Given the cost function, $C = C(Q)$, show that the marginal cost curve can only intersect the average cost curve where the slope of the AC curve is 0.

11. Individual A has a utility function of the form $u(c) = \alpha\sqrt{c}$ while individual B's utility is given by $U(C) = \beta \ln(C)$. Which of the two individuals is relatively more risk averse? Explain your answer.

    The formula for risk aversion is

$$\gamma = -\frac{u''(x).x}{u'(x)}$$

ADDITIONAL QUESTIONS

12. Show that in the case of a general 3x3 matrix, if matrix A is negative definite then matrix A is positive definite.

13. For each condition that needs to be met for a function to be continuous (see p.4 of Section 3) at some point x_0 sketch an example of a function that fails this condition, while the other conditions are satisfied (if possible).

TUTORIAL 3 SOLUTIONS

2010

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PART 1

1. Let the national income model be:

$$\begin{aligned} Y &= C + I + G \\ C &= a + b(Y - T) \qquad a > 0, 0 < b < 1 \\ T &= d + tY \qquad d > 0, 0 < t < 1 \end{aligned}$$

where Y is national income, C is (planned) consumption expenditure, I is investment expenditure, G is government expenditure and T is taxes.

(a) Identify which variables are endogenous, and which are exogenous.

(b) Give the economic meaning of the parameters a, b and g.

(c) Use Cramer's Rule to solve for the equilibrium national income, consumption and government expenditure.

(a) Endogenous: Y, C, T
Exogenous: I, G

(b) a is autonomous consumption
b is the marginal propensity to consume out of disposable income
d is non-income tax
t is the income tax rate

(c) We first need to re-arrange the equations so that the endogenous variables are on the LHS and the exogenous variables are on the RHS:

$$\begin{aligned} Y - C &= I + G \\ C - bY + bT &= a \\ T - tY &= d \end{aligned}$$

We can now rewrite this system in matrix form:

$$\mathbf{Ax} = \mathbf{b}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} I + G \\ a \\ d \end{bmatrix}$$

Calculate $|\mathbf{A}|$

$$\begin{aligned}
|\mathbf{A}| &= (1)\begin{vmatrix}1 & b\\ 0 & 1\end{vmatrix} - (-1)\begin{vmatrix}-b & b\\ -t & 1\end{vmatrix} + 0\\
&= (1)(1) + (1)(-b+bt)\\
&= 1 - b(1-t)\\
&\neq 0 \text{ since } 0<b<1 \text{ and } 0<t<1
\end{aligned}$$

Solve for Y^* :

$$\begin{aligned}
|\mathbf{A}_1| &= \begin{vmatrix}I+G & -1 & 0\\ a & 1 & b\\ d & 0 & 1\end{vmatrix}\\
&= a - bd + I + G
\end{aligned}$$

$$\begin{aligned}
Y^* &= \frac{|\mathbf{A}_1|}{|\mathbf{A}|}\\
&= \frac{a-bd+I+G}{1-b(1-t)}
\end{aligned}$$

Solve for C^* :

$$\begin{aligned}
|\mathbf{A}_2| &= \begin{vmatrix}1 & I+G & 0\\ -b & a & b\\ -t & d & 1\end{vmatrix}\\
&= a + bI + Gb - bd - btI - Gbt\\
&= a - bd + b(1-t)(I+G)
\end{aligned}$$

$$\begin{aligned}
C^* &= \frac{|\mathbf{A}_2|}{|\mathbf{A}|}\\
&= \frac{a-bd+b(1-t)(I+G)}{1-b(1-t)}
\end{aligned}$$

Solve for T^* :

$$\begin{aligned}
|\mathbf{A}_3| &= \begin{vmatrix}1 & -1 & I+G\\ -b & 1 & a\\ -t & 0 & d\end{vmatrix}\\
&= d + tI + Gt - bd + at\\
&= d - bd + at + t(I+G)
\end{aligned}$$

$$\begin{aligned} T^* &= \frac{|\mathbf{A}_3|}{|\mathbf{A}|} \\ &= \frac{d - bd + at + t\,(I + G)}{1 - b(1-t)} \end{aligned}$$

2. Determine the definiteness of the symmetric matrix

   $$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

   Let us first consider the leading principal minors:

   $$\begin{aligned} |\mathbf{M}_1| &= |3| = 3 > 0 \\ |\mathbf{M}_2| &= \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} = 2 > 0 \\ |\mathbf{M}_3| &= \begin{vmatrix} 3 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 6 \end{vmatrix} = -5 < 0 \end{aligned}$$

   The third leading principal minor is negative, therefore the matrix cannot be positive definite or positive semidefinite.

   Let us now consider the negative of the matrix:

   $$\begin{bmatrix} -3 & 1 & -1 \\ 1 & -1 & -2 \\ -1 & -2 & -6 \end{bmatrix}$$

   $$\begin{aligned} |\mathbf{M}_1| &= |-3| = -3 < 0 \\ |\mathbf{M}_2| &= \begin{vmatrix} -3 & 1 \\ 1 & -1 \end{vmatrix} = 2 > 0 \\ |\mathbf{M}_3| &= \begin{vmatrix} -3 & 1 & -1 \\ 1 & -1 & -2 \\ -1 & -2 & -6 \end{vmatrix} = 5 > 0 \end{aligned}$$

   The first leading principal minor is negative, so this matrix cannot be positive definite or positive semidefinite. This means that the original matrix is not negative definite or negative semidefinite.

   The matrix is *indefinite*.

3. For each of the following find $\frac{dy}{dx}$:

(a)

$$\begin{aligned} y &= \frac{x^3+2}{x-1} \\ \frac{dy}{dx} &= \frac{3x^2(x-1)-(x^3+2)(1)}{(x-1)^2} \quad \text{(using the quotient rule)} \\ &= \frac{3x^3-3x^2-x^3+2}{(x-1)^2} \\ &= \frac{2x^3-3x^2+2}{(x-1)^2} \end{aligned}$$

(b)

$$y = \frac{x^2\left(1-x^3\right)^2}{\sqrt[3]{4+x^2}}$$

Use logarithmic differentiation:
First take the natural log of both sides and simplify.

$$\begin{aligned} \ln y &= \ln x^2\left(1-x^3\right)^2 - \ln\left(4+x^2\right)^{\frac{1}{3}} \\ \ln y &= \ln x^2 + \ln\left(1-x^3\right)^2 - \ln\left(4+x^2\right)^{\frac{1}{3}} \\ \ln y &= 2\ln x + 2\ln\left(1-x^3\right) - \frac{1}{3}\ln\left(4+x^2\right) \end{aligned}$$

Now differentiate both sides w.r.t. x

LHS: Let $z = \ln y$ where $\ln y = 2\ln x + 2\ln\left(1-x^3\right) - \frac{1}{3}\ln\left(4+x^2\right)$

Using the chain rule: $\frac{dz}{dx} = \frac{dz}{dy}.\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$

RHS: $\frac{d}{dx} = \frac{2}{x} + 2\left(\frac{-3x^2}{1-x^3}\right) - \frac{1}{3}\left(\frac{2x}{4+x^2}\right)$

Now set LHS=RHS.

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{2}{x} + 2\left(\frac{-3x^2}{1-x^3}\right) - \frac{1}{3}\left(\frac{2x}{4+x^2}\right) \\ \frac{dy}{dx} &= y\left[\frac{2}{x} + 2\left(\frac{-3x^2}{1-x^3}\right) - \frac{1}{3}\left(\frac{2x}{4+x^2}\right)\right] \\ \frac{dy}{dx} &= \frac{x^2\left(1-x^3\right)^2}{\sqrt[3]{4+x^2}}\left[\frac{2}{x} + 2\left(\frac{-3x^2}{1-x^3}\right) - \frac{1}{3}\left(\frac{2x}{4+x^2}\right)\right] \\ \frac{dy}{dx} &= \frac{x^2\left(1-x^3\right)^2}{\sqrt[3]{4+x^2}}\left[\frac{2}{x} + \left(\frac{-6x^2}{1-x^3}\right) - \left(\frac{2x}{12+3x^2}\right)\right] \end{aligned}$$

(c)

$$\begin{aligned} y &= e^{4x^2+3x-1} \\ \frac{dy}{dx} &= (8x+3)e^{4x^2+3x-1} \end{aligned}$$

4. Find f_x and f_y:
   $f(x, y) = x^3 + 2xy - y^3$

$$\begin{aligned} f_x(x, y) &= \frac{\partial f}{\partial x} = 3x^2 + 2y \\ \text{and } f_y(x, y) &= \frac{\partial f}{\partial y} = 2x - 3y^2 \end{aligned}$$

5. The demand for a firm's output is given by:

$$Q = 50 - 0.25P - 0.2M + 0.10P_s$$

   where P is the price of the firm's good;

   M is the average weekly income of consumers;

   P_s is the price of the substitute good.

   (a) Demonstrate using derivatives what will happen to the demand for the firm's product if the firm raises the price for its good, or if the price of the substitute good rises.

   Here we simply differentiate the function with respect to the relevant parameter and interpret the result:
   So $\frac{\partial Q}{\partial P} = -0.25$ Therefore, if the firm was to raise its price, say by 1 unit, then the demand for its good will decrease by 0.25 of a unit.
   Similarly $\frac{\partial Q}{\partial P_s} = 0.10$ Thus, if the price of the substitute good was to increase, again by 1 unit, then the demand for the firm's good will increase by 0.10 of a unit.

   (b) Demonstrate, using derivatives, whether this is a normal or an inferior good. Motivate your answer.

Remember, if the quantity demanded increases as income increases then it is a normal good. If quantity demanded decreases as income increases then it is an inferior good.

So now $\frac{\partial Q}{\partial M} = -0.2$ Therefore, as income (M) increases quantity demanded decreases. Thus, the good is .... an inferior good.

(c) If we assume that $P = 12$, $M = 100$, and $P_s = 30$, what is the price elasticity of demand for the firm's product?

Recall the following

$$\varepsilon_{Q.P} = \frac{dQ/dP}{Q/P}$$

Now $\frac{dQ}{dP} = -0.25$

And $\frac{Q}{P} = \frac{50 - 0.25P - 0.2M + 0.1P_s}{P}$

Substitute in our given values and we get $\frac{Q}{P} = 2.5$

So

$$\begin{aligned}\varepsilon_{Q.P} &= \frac{-0.25}{2.5} \\ &= -0.1\end{aligned}$$

So demand is relatively inelastic.

$$|\varepsilon_{Q.P}| = 0.1$$

6. Individual A has a utility function of the form $u(c) = 4\alpha\gamma\sqrt{c}$ while individual B's utility is given by $U(C) = 5\beta\ln(C)$, where c, C represent A and B's consumption respectively. Show that the preferences of these two individuals' exhibit *local non-satiation* and *diminishing marginal utility* of consumption.

To show *non-satiation*, you need to show that the first derivative is positive.

Therefore, for A we get the following:

$$\begin{aligned}u'(c) &= 2\alpha\gamma c^{-\frac{1}{2}} \\ &= \frac{2\alpha\gamma}{\sqrt{c}} > 0 \text{ if } \alpha, \gamma > 0\end{aligned}$$

And for B we get:

$$U'(C) = \frac{5\beta}{C} > 0 \text{ if } \beta > 0$$

To show *diminishing marginal utility* we need to show that the second derivative is negative.

So, for A we get:

$$\begin{aligned} u''(c) &= -\alpha\gamma c^{-\frac{3}{2}} \\ &= \frac{-\alpha\gamma}{\sqrt{c^3}} < 0 \end{aligned}$$

And for B we get:

$$U''(C) = \frac{-5\beta}{C^2} < 0$$

PART 2

7. Determine the definiteness of the symmetric matrices

   (a) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 9 \end{bmatrix}$

   Let us first consider the leading principal minors:

$$\begin{aligned} |\mathbf{M}_1| &= |3| = 3 > 0 \\ |\mathbf{M}_2| &= \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} = 2 > 0 \\ |\mathbf{M}_3| &= \begin{vmatrix} 3 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 9 \end{vmatrix} = 1 > 0 \end{aligned}$$

   The leading principal minors are all positive, therefore the matrix is *positive definite*.

   (b) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Let us first consider the leading principal minors:

$$\begin{aligned}
|\mathbf{M}_1| &= |-1| = -1 < 0 \\
|\mathbf{M}_2| &= \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = 1 > 0 \\
|\mathbf{M}_3| &= \begin{vmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{vmatrix} = 0
\end{aligned}$$

The first leading principal minor is negative, therefore the matrix cannot be positive definite or positive semidefinite.
Let us now consider the negative of the matrix:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{aligned}
|\mathbf{M}_1| &= |1| = 1 > 0 \\
|\mathbf{M}_2| &= \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1 > 0 \\
|\mathbf{M}_3| &= \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix} = 0
\end{aligned}$$

The third leading principal minor is zero so this matrix cannot be positive definite. To determine whether it is positive semidefinite, we must consider the other principal minors:

$$\begin{aligned}
|2| &= 2 > 0 \\
|1| &= 1 > 0 \\
\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} &= 1 > 0 \\
\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} &= 1 > 0
\end{aligned}$$

All the principal minors (including the leading ones) are non-negative so this matrix is positive semidefinite. This means that the original matrix is *negative semidefinite*.

8. For each of the following find $\frac{dy}{dx}$:

(a)

$$\begin{aligned} y &= (x+7)(2x+3) \\ \frac{dy}{dx} &= 1(2x+3)+2(x+7) \quad \text{(using the product rule)} \\ &= 4x+17 \end{aligned}$$

(b)

$$\begin{aligned} y &= (x^2-2x+1)^{15} \\ \frac{dy}{dx} &= 15(x^2-2x+1)^{14}(2x-2) \quad \text{(using the chain rule)} \end{aligned}$$

If it helps you let

$$\begin{aligned} (x^2-2x+1) &= z \\ \implies y &= z^{15} \\ \text{so then } \frac{dy}{dx} &= \frac{dy}{dz}.\frac{dz}{dx} \end{aligned}$$

(c)

$$\begin{aligned} y &= \ln 3x + \ln(5+x^2) \\ \frac{dy}{dx} &= \frac{3}{3x} + \frac{2x}{5+x^2} \\ &= \frac{1}{x} + \frac{2x}{5+x^2} \\ &= \frac{5+x^2+2x^2}{x(5+x^2)} \\ &= \frac{5+3x^2}{x(5+x^2)} \end{aligned}$$

9. Find f_x and f_y:

$z = f(x,y) = \frac{3x-2y}{x^2-y}$

$$\begin{aligned}
\text{Then} \quad f_x(x,y) &= \frac{3(x^2-y)-2x(3x-2y)}{(x^2-y)^2} \text{ (applying the quotient rule)} \\
&= \frac{-3x^2-3y+4xy}{(x^2-y)^2} \\
\text{Likewise} \quad f_y(x,y) &= \frac{-2(x^2-y)-(-1)(3x-2y)}{(x^2-y)^2} \\
&= \frac{-2x^2+3x}{(x^2-y)^2} \\
&= \frac{x(-2x+3)}{(x^2-y)^2}
\end{aligned}$$

10. Given the cost function, $C = C(Q)$, show that the marginal cost curve can only intersect the average cost curve where the slope of the AC curve is 0.

    HINT: Look for key words which will help you figure out what functions to solve for.

    In this case you will need an expression for MC and a separate expression the slope of the AC curve.

    Start with MC: As you should know MC is just the first derivative of the TC function where $TC = C = C(Q)$. Hence

$$MC = \frac{dTC}{dQ} = C'(Q)$$

    Now you need to find the *slope* of the ACcurve. To get this, you will need to get an expression for AC, and then take the first derivative. So

$$AC = \frac{TC}{Q} = \frac{C(Q)}{Q}$$

$$\therefore \frac{dAC}{dQ} = \frac{C'(Q)Q - 1C(Q)}{Q^2} \quad \text{(using the quotient rule)}$$

    Now the question asks you to show that MC intersects AC (i.e. they are equal to each other) when the slope of AC is zero. So, take your expression for the slope of the AC curve (i.e. $\frac{dAC}{dQ}$) and set it equal to zero and solve for AC.

$$
\begin{aligned}
\text{slope} = \frac{C'(Q)Q - 1C(Q)}{Q^2} &= 0 \\
C'(Q)Q - 1C(Q) &= 0 \qquad \text{multiplying by } Q^2 \text{ to simplify} \\
\Rightarrow C'(Q)Q &= C(Q) \\
\Rightarrow C'(Q) &= \frac{C(Q)}{Q} \quad \text{dividing by } Q \text{ which is fine since } Q \text{ is positive} \\
\Rightarrow MC &= AC
\end{aligned}
$$

Thus you have found that $MC = AC$ by setting $\frac{dAC}{dQ} = 0.$, i.e. you have proved what you need to.

11. Individual A has a utility function of the form $u(c) = \alpha\sqrt{c}$ while individual B's utility is given by $U(C) = \beta \ln(C)$. Which of the two individuals is relatively more risk averse? Explain your answer.

The formula for risk aversion is

$$\gamma = -\frac{u''(x).x}{u'(x)}$$

So, for A:

$$
\begin{aligned}
\gamma_A &= -\left[\frac{\left(-\frac{1}{4}\alpha c^{-\frac{3}{2}}\right) \times c}{\left(\frac{1}{2}\alpha c^{-\frac{1}{2}}\right)}\right] \\
&= -\left[\frac{-\frac{1}{4}\alpha c^{-\frac{1}{2}}}{\frac{1}{2}\alpha c^{-\frac{1}{2}}}\right] \\
&= \frac{1}{2}
\end{aligned}
$$

Similarly, for B:

$$
\begin{aligned}
\gamma_B &= -\left[\frac{(-\beta/C^2) \times C}{(\beta/C)}\right] \\
&= 1
\end{aligned}
$$

Therefore, individual B is relatively more risk averse than individual A because B's coefficient for risk aversion is higher.