Quantitative Methods for Economics

Tutorial 1

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TUTORIAL 1

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ECO3021S

PART 1

1. In the following paired statements, let p be the first statement and q the second. Indicate for each case whether $p \Rightarrow q, p \Leftarrow q$ or $p \Leftrightarrow q$.

   (a) It is a holiday; it is Christmas Day.
   (b) The geometric figure has four sides; it is a rectangle.
   (c) Two ordered pairs (a, b) and (b, a) are equal; a is equal to b.
   (d) $x \in \mathbb{Z}$; $x \in \mathbb{R}$.
   (e) The petrol tank in my car is empty; I cannot start my car.

2. Simplify the following expressions:

   (a) $x^2 \times x^0$
   (b) $a^3b^2a^5b$
   (c) $\left(\dfrac{x^{1/5}y^{6/5}}{z^{2/5}}\right)^5$
   (d) $\ln e^{ab} + \ln e^{3a}$
   (e) $e^{\ln x^2}$

3. A firm's output Y is related to capital input K, labour input L and natural resource input R by the production function

$$Y = 2K^{1/2}L^{1/3}R^{1/6}$$

   Write down a linear relationship between the logarithms of Y, K, L, R.

4. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

   Show that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

5. Let

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 3 & -2 \\ 4 & 2 \\ 5 & -3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 & 1 & 2 \\ -1 & 0 & -2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 3 & -1 & 4 \\ -4 & 2 & 1 & 0 \end{bmatrix}$$

Find

(a) **AB**
(b) **CA**

6. A firm with 5 retails stores stocks TVs (t), hi-fis (h), DVD players (d) and computers (c).

   Shop 1 has $10t, 15h, 9d$ and $12c$.

   Shop 2 has $20t, 14h, 8d$ and $5c$.

   Shop 3 has $16t, 8h, 15d$ and $6c$.

   Shop 4 has $25t, 15h, 7d$ and $16\dot{c}$.

   Shop 5 has $5t, 12h, 20d$ and $18c$.

   Express the current inventory in the form of a matrix.

7. Solve the following system of equations $\mathbf{Ax} = \mathbf{b}.$

$$\begin{bmatrix} 4 & 1 & -4 & 6 \\ -5 & 5 & 0 & 3 \\ -4 & -4 & 10 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

8. Show in the 3x3 case that there exists only one matrix I that satisfies $AI = IA = A$, for any matrix A, and that is the (3x3) identity matrix.

PART 2

9. Simplify the following expressions:

   (a) $\dfrac{x^3}{x^{-3}}$

   (b) $\dfrac{x^2y^7}{x^3y^5}$

   (c) $\left(x^{-1} - y^{-1}\right)^{-2}$

   (d) $\ln Ae^4 + \ln e^7$

   (e) $\ln e + \log \dfrac{1}{10}$

   (f) $\ln \dfrac{x^2}{(x+1)^3}$

(g) $\ln \dfrac{\sqrt{x}}{(x+1)^2 (x+2)^3}$

(h) $\log_2(2x) - \log_2(x+1)$

10. The components of the 3-vector $\mathbf{a}$ are Vuyo's weekly expenditures on food, clothing and housing. The components of the 3-vector $\mathbf{b}$ are Siyanda's weekly expenditures on food, clothing and housing. Interpret the vectors $\mathbf{a}+\mathbf{b}$ and $52\mathbf{a}$.

11. If

$$\mathbf{a} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

find

(a) $\mathbf{a}+\mathbf{b}$

(b) $4\mathbf{a}$

(c) $-6\mathbf{b}$

(d) $2\mathbf{a} - 3\mathbf{b}$

12. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(a) Find $-2\mathbf{B}$.

(b) Find $\mathbf{AB}$.

13. Let

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 3 & -2 \\ 4 & 2 \\ 5 & -3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 & 1 & 2 \\ -1 & 0 & -2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 3 & -1 & 4 \\ -4 & 2 & 1 & 0 \end{bmatrix}$$

Find

(a) $\mathbf{BA}$

(b) $\mathbf{AC}$

14. A clothing store discounts all its jeans, jackets and suits by 20% at the end of the season. If $\mathbf{V}_1$ is the value of the stock in its three branches prior to the discount, find the value of $\mathbf{V}_2$ after the discount where

$$\mathbf{V}_1 = \begin{bmatrix} 5000 & 4500 & 6000 \\ 10000 & 12000 & 7500 \\ 8000 & 9000 & 11000 \end{bmatrix}$$

15. Solve the following systems of equations $\mathbf{Ax} = \mathbf{b}.$

(a)

$$\begin{bmatrix} 1 & 2 & 4 & -1 \\ 0 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$$

ADDITIONAL QUESTIONS

16. Prove the following

(a) $x^m \times x^n = x^{m+n}$

(b) $(x^m)^n = x^{mn}$

(c) $\log_a \left(y^b\right) = b \times \log_a y$

TUTORIAL 1 SOLUTIONS

2010

ECO3021S

1. In the following paired statements, let p be the first statement and q the second. Indicate for each case whether $p \Rightarrow q, p \Leftarrow q$ or $p \Leftrightarrow q$.

   (a) It is a holiday; it is Christmas Day.

   In order for it to be a holiday, it is not necessary that it is Christmas day, so $p \nRightarrow q$.

   If it is Christmas Day, it is sufficent to conclude that it is a holiday, so $p \Leftarrow q$.

   Thus q is sufficient, but not necessary, for p, i.e. $p \Leftarrow q$.

   (b) The geometric figure has four sides; it is a rectangle.

   In order for the geometric figure to have four sides, it is not necessary that it is a rectangle, so $p \nRightarrow q$.

   If it is a rectangle, it is sufficent to conclude that the geometric figure has four sides, so $p \Leftarrow q$.

   Thus q is sufficient, but not necessary, for p, i.e. $p \Leftarrow q$.

   (c) Two ordered pairs (a, b) and (b, a) are equal; a is equal to b.

   In order for two ordered pairs (a, b) and (b, a) to be equal, it is necessary that a is equal to b, so $p \Rightarrow q$.

   If a is equal to b, it is sufficent to conclude that the two ordered pairs (a, b) and (b, a) are equal, so $p \Leftarrow q$.

   Thus q is necessary and sufficient for p, i.e. $p \Leftrightarrow q$.

   (d) $x \in \mathbb{Z}$; $x \in \mathbb{R}$.

   In order for $x \in \mathbb{Z}$ (i.e. x is an integer), it is necessary that $x \in \mathbb{R}$ (i.e. x is a real number), so $p \Rightarrow q$.

   If $x \in \mathbb{R}$ (i.e. x is a real number), it is not sufficent to conclude that $x \in \mathbb{Z}$ (i.e. x is an integer), so $p \nLeftarrow q$.

   Thus q is necessary, but not sufficient, for p, i.e. $p \Rightarrow q$.

   (e) The petrol tank in my car is empty; I cannot start my car.

   In order for the petrol tank in my car to be empty, it is necessary that I cannot start my car, so $p \Rightarrow q$.

   If I cannot start my car, it is not sufficent to conclude that the petrol tank in my car is empty, so $p \nLeftarrow q$.

   Thus q is necessary, but not sufficient, for p, i.e. $p \Rightarrow q$.

2. Simplify the following expressions:

   (a) $x^2 \times x^0 = x^2$
   (b) $a^3b^2a^5b = a^8b^3$
   (c) $\left(\frac{x^{1/5}y^{6/5}}{z^{2/5}}\right)^5 = \frac{xy^6}{z^2}$
   (d) $\ln e^{ab} + \ln e^{3a} = ab + 3a$
   (e) $e^{\ln x^2} = x^2$

3. A firm's output Y is related to capital input K, labour input L and natural resource input by the production function

$$Y = 2K^{1/2}L^{1/3}R^{1/6}$$

   Write down a linear relationship between the logarithms of Y, K, L, R.

$$\ln Y = \ln 2 + \frac{1}{2}\ln K + \frac{1}{3}\ln L + \frac{1}{6}\ln R$$

4. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

   Show that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} = \begin{bmatrix} 4 & 1 & 7 \\ 3 & 3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

5. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

   (a)

$$\mathbf{AB} = \begin{bmatrix} -4 & -1 & -2 \\ 14 & 3 & 10 \\ 14 & 4 & 4 \\ 23 & 5 & 16 \end{bmatrix}$$

(b)

$$\mathbf{CA} = \begin{bmatrix} 24 & -20 \\ 14 & -2 \end{bmatrix}$$

6. A firm with 5 retails stores stocks TVs (t), hi-fis (h), DVD players (d) and computers (c).

   Shop 1 has $10t, 15h, 9d$ and $12c$.

   Shop 2 has $20t, 14h, 8d$ and $5c$.

   Shop 3 has $16t, 8h, 15d$ and $6c$.

   Shop 4 has $25t, 15h, 7d$ and $16\dot{c}$.

   Shop 5 has $5t, 12h, 20d$ and $18c$.

   Express the current inventory in the form of a matrix.

$$\begin{array}{cc} & \begin{array}{cccc} t & h & d & c \end{array} \\ \text{Shop number} & \begin{bmatrix} 10 & 15 & 9 & 12 \\ 20 & 14 & 8 & 5 \\ 16 & 8 & 15 & 6 \\ 25 & 15 & 7 & 16 \\ 5 & 12 & 20 & 18 \end{bmatrix} \end{array}$$

7. Solve the following system of equations $\mathbf{Ax} = \mathbf{b}.$

$$\begin{bmatrix} 4 & 1 & -4 & 6 \\ -5 & 5 & 0 & 3 \\ -4 & -4 & 10 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

   This system has no solution. The last equation reduces to $0 = 4\,(!)$.

8. Show in the 3x3 case that there exists only one matrix I that satisfies $AI = IA = A$, for any matrix A, and that is the (3x3) identity matrix.

   See accompanying file with answer to this question.

Part 2

9. Simplify the following expressions:

   (a) $\dfrac{x^3}{x^{-3}} = x^6$

   (b) $\dfrac{x^2y^7}{x^3y^5} = \dfrac{y^2}{x}$

   (c) $\left(x^{-1} - y^{-1}\right)^{-2} = \dfrac{x^2y^2}{(y-x)^2} = \left(\dfrac{1}{x} - \dfrac{1}{y}\right)^{-2} = \left(\dfrac{y-x}{xy}\right)^{-2}$ etc.

   (d) $\ln Ae^4 + \ln e^7 = \ln A + 11$

   (e) $\ln e + \log \dfrac{1}{10} = 1 - 1 = 0$

   (f) $\ln \dfrac{x^2}{(x+1)^3} = 2\ln x - 3\ln(x+1)$

   (g) $\ln \dfrac{\sqrt{x}}{(x+1)^2 (x+2)^3} = \dfrac{1}{2}\ln x - 2\ln(x+1) - 3\ln(x+2)$

   (h) $\log_2(2x) - \log_2(x+1) = \log_2 \dfrac{2x}{x+1}$

10. The components of the 3-vector **a** are Vuyo's weekly expenditures on food, clothing and housing. The components of the 3-vector **b** are Siyanda's weekly expenditures on food, clothing and housing. Interpret the vectors $\mathbf{a} + \mathbf{b}$ and $52\mathbf{a}$.

    Components of $\mathbf{a} + \mathbf{b}$ are the sums of Vuyo's and Ayanda's weekly expenditures on food, clothing and housing; components of $52\mathbf{a}$ are Vuyo's annual expenditures on food, clothing and housing.

11. If

$$\mathbf{a} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

find

   (a)

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

   (b)

$$4\mathbf{a} = \begin{bmatrix} 8 \\ 20 \end{bmatrix}$$

(c)

$$-6\mathbf{b} = \begin{bmatrix} -42 \\ -6 \end{bmatrix}$$

(d)

$$2\mathbf{a} - 3\mathbf{b} = \begin{bmatrix} 4 \\ 10 \end{bmatrix} - \begin{bmatrix} 21 \\ 3 \end{bmatrix} = \begin{bmatrix} -17 \\ 7 \end{bmatrix}$$

12. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(a) Find $-2\mathbf{B}$.

$$-2\mathbf{B} = \begin{bmatrix} -6 & -2 & -4 \\ 0 & -2 & -2 \\ 0 & -2 & 0 \end{bmatrix}$$

(b) Find $\mathbf{AB}$.

$$\mathbf{AB} = \begin{bmatrix} 3 & 6 & 2 \\ 9 & 6 & 8 \\ 6 & 3 & 5 \end{bmatrix}$$

13. Let

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 3 & -2 \\ 4 & 2 \\ 5 & -3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 & 1 & 2 \\ -1 & 0 & -2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 3 & -1 & 4 \\ -4 & 2 & 1 & 0 \end{bmatrix}$$

Find

(a) $\mathbf{BA}$ is not defined because the matrices are not conformable.

(b)

$$\mathbf{AC} = \begin{bmatrix} -1 & -3 & 1 & -4 \\ 11 & 5 & -5 & 12 \\ -4 & 16 & -2 & 16 \\ 17 & 9 & -15 & 20 \end{bmatrix}$$

14. A clothing store discounts all its jeans, jackets and suits by 20% at the end of the season. If $\mathbf{V}_1$ is the value of the stock in its three branches prior to the discount, find the value of $\mathbf{V}_2$ after the discount where

$$\mathbf{V}_1 = \begin{bmatrix} 5000 & 4500 & 6000 \\ 10000 & 12000 & 7500 \\ 8000 & 9000 & 11000 \end{bmatrix}$$

$$\begin{aligned} \mathbf{V}_2 &= 0.8 \begin{bmatrix} 5000 & 4500 & 6000 \\ 10000 & 12000 & 7500 \\ 8000 & 9000 & 11000 \end{bmatrix} \\ &= \begin{bmatrix} 4000 & 3600 & 4800 \\ 8000 & 9600 & 6000 \\ 6400 & 7200 & 8800 \end{bmatrix} \end{aligned}$$

15. Solve the following systems of equations $\mathbf{Ax} = \mathbf{b}.$

(a)

$$\begin{bmatrix} 1 & 2 & 4 & -1 \\ 0 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

This is in row echelon form. Columns 1 and 3 are basic, so we assign arbitrary values to x_2 and x_4. Let $x_2 = \lambda$ and $x_4 = \mu$.Then

$$\begin{aligned} x_1 + 4x_3 &= -2\lambda + \mu + 5 \\ 2x_3 &= -3\mu + 6 \end{aligned}$$

Solve for x_1 and x_3 by back substitution:

$$\begin{aligned} x_3 &= -\frac{3}{2}\mu + 3 \\ x_1 &= -2\lambda + \mu - 4\left(-\frac{3}{2}\mu + 3\right) + 5 = -2\lambda + 7\mu - 7 \end{aligned}$$

The complete solution in vector form is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 7 \\ 0 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

There are infinitely many solutions to this system.

(b)

$$\begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$$

The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right] \begin{array}{l} \\ (2) - 3 \times (1) \\ (3) - 2 \times (1) \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right] (2) \times \tfrac{-1}{4}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 7 & -1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} (1) - 3 \times (2) \\ \\ (3) + 2 \times (2) \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 15 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} (1) - 7 \times (3) \\ (2) + 3 \times (3) \end{array}$$

The complete solution in vector form is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -15 \\ 8 \\ 2 \end{bmatrix}$$

TUTORIAL 1, QUESTION 8

There are basically two steps to this:

1. Show that $AI = A$ and $IA = A$ for any (3×3) matrix A
   and $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. Show that there is no <u>other</u> matrix, J, that satisfies
   $JA = A, \quad AJ = A$ for any A
   ('other' meaning $J \neq I$)

Step 1

$AI = A$:
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (a_{11}.1 + a_{12}.0 + a_{13}.0) & \text{etc.} \\ (a_{21}.1 + a_{22}.0 + a_{23}.0) & \text{etc.} \\ \text{etc.} & \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A$$

and similarly for $IA = A$

Step 2 We show that any J satisfying $JA = AJ = A$ equals I.

Assume there <u>is</u> some J that satisfies $JA = A$, for any A

Let $A = I$ (we can chose any matrix after all...)

then $JI = I$

But we know $AI = A$ for any A (see step 1)

so we must have $JI = J$

$\therefore \quad I = JI = J$

i.e. There is no matrix not equal to I that satisfies the requirements for an identity matrix [Note: More formally written, this would be a 'proof by contradiction']