# Mathematics for Economists

## Tutorial Questions – Optimisation

# Tutorial 3: Optimisation

ECO4112F 2011

1. Find and classify the extrema of the cubic cost function $C = C(Q) = 2Q^3 - 5Q^2 + 2$.

2. Find and classify the extreme values of the following functions

   (a) $y = x_1^2 + 3x_2^2 - 3x_1x_2 + 4x_2x_3 + 6x_3^2$

   (b) $y = x_1x_3 + x_1^2 - x_2 + x_2x_3 + x_2^2 + 3x_3^2$

3. State whether each of the following functions are concave, strictly concave, convex, strictly convex or neither:

   (a) $y = x^2$

   (b) $y = 2x^2 - xz + z^2$

   (c) $y = -x^2$

   (d) $y = -xz$

4. Find the extreme values of the function $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$ and show that only one of the two extrema can be characterised as a local minimum and neither of the extrema can be characterised as a local maximum. If $x^* = 0$, how would you characterise the curvature of $z$ at $x^*$?

5. Identify the stationary point of the function

$$f(x_1, x_2, x_3) = 2x_1^2 - 21x_1 - 3x_1x_2 + 3x_2^2 - 2x_2x_3 + x_3^2$$

   and show that this stationary point is a unique global minimum.

6. Identify the stationary point of the function

$$f(x_1, x_2, x_3) = 2x_1x_2 - \frac{1}{2}x_1^2 - 3x_2^2 + x_2x_3 - 1.5x_3^2 + 10x_3$$

   and determine if it represents a maximum or minimum.

7. Find the extremum of the following functions

   (a) $z = xy$ subject to $x + 2y = 2$

   (b) $z = x - 3y - xy$ subject to $x + y = 6$

   (c) $y = x^2 + 2xz + 4z^2$ subject to $x + z = 8$

   (d) $y = 3x^2 + z^2 - 2xz$ subject to $x + z = 1$

   (e) $y = \ln 2x + 2\ln z$ subject to $z = 16 - 4x$

8. Find the optimal values of each of the choice variables in the functions below, and solve for the Lagrangian multiplier too.

   (a) $U(x, z) = x^2 + 2x + 3z^2 - 6z + x$ subject to $2x + 2z = 32$.
   (b) $U(a, b, c) = \frac{1}{2}a^2 + 4b^2 - 4a + 8c^2$ subject to $\frac{1}{2}a + 3b + c$.

9. Let $A, \alpha, \beta$ be positive constants. Show that the Cobb-Douglas production function

$$Q = AK^{\alpha}L^{\beta}$$

   is concave if and only if $\alpha + \beta \leq 1$.

10. A monopolist can produce quantities $x$ and $y$ of two products $X$ and $Y$ at cost $4x^2 + xy + 2y^2$. The inverse demand functions are

$$\begin{aligned} p_X &= 150 - 5x + y \\ p_Y &= 30 + 2x - 2y \end{aligned}$$

   where $p_X$ and $p_Y$ are the prices charged for $X$ and $Y$.

   (a) Find the values of $x, y, p_X$ and $p_Y$ which maximise profit, and the maximal profit.
   (b) Confirm that your answer to (a) represents a maximum.

11. Find the extrema of

$$z = -2x^2 - 21x + 8xy + y^3 - 2y^2 + 6y$$

   and classify the extrema as minima or maxima, and as global (unique or not unique) or local.

12. Billy has a Cobb-Douglas utility function

$$U(x_1, x_2) = x_1^{3/4}x_2^{1/4}$$

   where $x_1, x_2$ denote the consumption of cars and jets respectively. The prices of the goods are $p_1, p_2$ and Billy's income is $m$.

   (a) Find the demand functions for cars and jets that maximise Billy's utility.
   (b) Confirm that your answer to (a) represents a constrained maximum.
   (c) What is the economic interpretation of the Lagrangian multiplier in this case?

13. The production of Jimmy Choo shoes ($Q$) requires two inputs, capital ($K$) and labour ($L$):

$$Q = f(K, L) = 9K^{1/3}L^{2/3}$$

A unit of capital costs R4 and a unit of labour costs R16.

(a) Does the production function exhibit increasing, decreasing or constant returns to scale? Explain your answer.

(b) Show that the production function exhibits diminishing returns to each factor of production.

(c) What is the cost minimising bundle of capital and labour to produce any *fixed* $Q = \overline{Q}$?

(d) Confirm that your answer to (a) represents a constrained minimum.

(e) What is the economic interpretation of the Lagrangian multiplier in this case?

14. Solve the following maximisation problem by applying the Kuhn-Tucker theorem:

$$\max_{x,y} \; -4x^2 - 2xy - 4y^2$$

$$\begin{aligned} \text{subject to } x + 2y &\leq 2 \\ 2x - y &\leq -1 \end{aligned}$$

15. Solve the following maximisation problem by applying the Kuhn-Tucker theorem:

$$\max_{x,y} \; 3.6x - 0.4x^2 + 1.6y - 0.2y^2$$

$$\begin{aligned} \text{subject to } 2x + y &\leq 10 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

16. *Consider the issue of optimal time allocation*:

Suppose that you have a utility function defined over two goods, $x_1$ and $x_2$, such that your utility function is $u(x_1, x_2)$.

Suppose it takes $t_1$ units of time to consume $x_1$ units of Swiss chocolate and $t_2$ units of time to consume $x_2$ units of champagne.

The total time available for consumption is $T$.

Your challenge is to maximize your utility subject to your income constraint $p_1x_1 + p_2x_2 \leq m$ and time constraint $t_1x_1 + t_2x_2 \leq T$.

Assume that both $x_1$ and $x_2$ are strictly positive at the optimum and $u_1$ and $u_2$ are positive, reflecting non-satiation.

(a) Set up a Lagrangian function in which utility is maximized subject to both the time and income constraint and specify the first order conditions for the optimum.

(b) Consider the four solution possibilities to this problem based on the four possible ways in which the complementary slackness conditions might be met. Which is most likely to yield the optimal solution?

$$\begin{aligned}
\lambda &> 0 \text{ and } \mu > 0 \\
\lambda &> 0 \text{ and } \mu = 0 \\
\lambda &= 0 \text{ and } \mu > 0 \\
\lambda &= 0 \text{ and } \mu = 0
\end{aligned}$$

# Tutorial 3: Optimisation

## SELECTED SOLUTIONS

ECO4112F 2011

1. $Q = 0$ is a maximum

   $Q = \frac{10}{6}$ is a minimum

2.
   (a) $y = 0$ : minimum

   (b) $y = -\frac{11}{40}$ : minimum

3.
   (a) strictly convex

   (b) strictly convex

   (c) strictly concave

   (d) neither

4. FOC

$$\begin{aligned} z_x &= 24x^2 + 2y - 6x = 0 \\ z_y &= 2x + 2y = 0 \end{aligned}$$

Solve simultaneously to get the extrema: $x^* = y^* = 0$ and $x^* = y^* = \frac{1}{3}$.

SOC:

$$\mathbf{H} = \begin{bmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{bmatrix} = \begin{bmatrix} 48x - 6 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\begin{aligned} |\mathbf{H}_1| = |48x - 6| = 48x - 6 &< 0 \text{ when } x < \frac{6}{48} \\ &> 0 \text{ when } x > \frac{6}{48} \end{aligned}$$

Thus, the Hessian cannot be everywhere positive or negative definite and so neither of our extrema can be global.

We evaluate the Hessian at each critical point. Consider $x^* = y^* = 0$ :

$$\mathbf{H} = \begin{bmatrix} -6 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\begin{aligned} |\mathbf{H}_1| &= |-6| = -6 < 0 \\ |\mathbf{H}_2| &= |\mathbf{H}| = -16 < 0 \end{aligned}$$

Thus, $\mathbf{H}$ is indefinite at $x^* = y^* = 0$.

Consider $x^* = y^* = \frac{1}{3}$ :

$$\mathbf{H} = \begin{bmatrix} 10 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\begin{aligned} |\mathbf{H}_1| &= |10| = 10 > 0 \\ |\mathbf{H}_2| &= |\mathbf{H}| = 16 > 0 \end{aligned}$$

Thus, $\mathbf{H}$ is positive definite at $x^* = y^* = \frac{1}{3}$ and so this point is a local minimum.

5. FOC:

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 4x_1 - 21 - 3x_2 = 0 \\ \frac{\partial f}{\partial x_2} &= -3x_1 + 6x_2 - 2x_3 = 0 \\ \frac{\partial f}{\partial x_3} &= -2x_2 + 2x_3 = 0 \end{aligned}$$

Solve these simultaneously to get the stationary point.

SOC:

$$\mathbf{H} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} 4 & -3 & 0 \\ -3 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$\begin{aligned} |\mathbf{H}_1| &= |4| = 4 > 0 \\ |\mathbf{H}_2| &= \begin{vmatrix} 4 & -3 \\ -3 & 6 \end{vmatrix} = 15 > 0 \\ |\mathbf{H}_3| &= |\mathbf{H}| = 46 > 0 \end{aligned}$$

The Hessian matrix is everywhere positive definite, and so the stationary point represents a unique global minimum.

6. $x_1^* = 4, x_2^* = 2, x_3^* = 4$. This is a unique global maximum.

7.

(a) $z^* = \frac{1}{2}$ when $\lambda^* = \frac{1}{2}, x^* = 1, y^* = \frac{1}{2}$.
$\left|\overline{\mathbf{H}}_2\right| = 4 > 0$ and so this is a constrained maximum.

(b) $z^* = -19$ when $\lambda^* = -4, x^* = 1, y^* = 5$.
$\left|\overline{\mathbf{H}}_2\right| = -2 < 0$ and so this is a constrained minimum.

(c) $(x^*, z^*) = (8, 0)$ Constrained minimum

(d) $(x^*, z^*) = \left(\frac{1}{3}, \frac{2}{3}\right)$ Constrained minimum

(e) $(x^*, z^*) = \left(\frac{4}{3}, \frac{32}{3}\right)$ Constrained maximum

8.

(a) $(x^*, z^*, \lambda^*) = (12, 4, 15)$

(b) $(a^*, b^*, c^*, \lambda^*) = (12, 6, 1, 16)$

9. First order derivatives:

$$\begin{aligned} \frac{\partial Q}{\partial K} &= \alpha A K^{\alpha-1} L^{\beta} \\ \frac{\partial Q}{\partial L} &= \beta A K^{\alpha} L^{\beta-1} \end{aligned}$$

Hessian:

$$\begin{aligned} \mathbf{H} &= \begin{bmatrix} \frac{\partial^2 Q}{\partial K^2} & \frac{\partial Q}{\partial K}\frac{\partial Q}{\partial L} \\ \frac{\partial Q}{\partial L}\frac{\partial Q}{\partial K} & \frac{\partial^2 Q}{\partial L^2} \end{bmatrix} \\ &= \begin{bmatrix} \alpha(\alpha-1) A K^{\alpha-2} L^{\beta} & \alpha\beta A K^{\alpha-1} L^{\beta-1} \\ \alpha\beta A K^{\alpha-1} L^{\beta-1} & \beta(\beta-1) A K^{\alpha} L^{\beta-2} \end{bmatrix} \end{aligned}$$

For production function to be concave, the Hessian must be negative definite, i.e. the signs of the principal minors must alternate in sign, starting with a non-positive.

$$\begin{aligned} |\mathbf{H}_1| &= \alpha(\alpha-1) A K^{\alpha-2} L^{\beta} \\ \text{Since } A,\ K \text{ and } L \text{ are positive} & \\ \alpha(\alpha-1) &\leq 0 \\ &\Rightarrow 0 \leq \alpha \leq 1 \end{aligned}$$

$$
\begin{aligned}
|\mathbf{H}_2| &= \begin{vmatrix} \alpha(\alpha-1)AK^{\alpha-2}L^{\beta} & \alpha\beta AK^{\alpha-1}L^{\beta-1} \\ \alpha\beta AK^{\alpha-1}L^{\beta-1} & \beta(\beta-1)AK^{\alpha}L^{\beta-2} \end{vmatrix} \\
&= \left[\alpha(\alpha-1)AK^{\alpha-2}L^{\beta}\right]\left[\beta(\beta-1)AK^{\alpha}L^{\beta-2}\right] - \left[\alpha\beta AK^{\alpha-1}L^{\beta-1}\right]\left[\alpha\beta AK^{\alpha-1}L^{\beta-1}\right] \\
&= \alpha(\alpha-1)\beta(\beta-1)A^2K^{2\alpha-2}L^{2\beta-2} - \alpha^2\beta^2A^2K^{2\alpha-2}L^{2\beta-2} \\
&= A^2K^{2\alpha-2}L^{2\beta-2}\left[\alpha(\alpha-1)\beta(\beta-1) - \alpha^2\beta^2\right]
\end{aligned}
$$

Since $A$, $K$ and $L$ are positive,

$$
\begin{aligned}
\alpha(\alpha-1)\beta(\beta-1) - \alpha^2\beta^2 &\geq 0 \\
\left(\alpha^2-\alpha\right)\left(\beta^2-\beta\right) - \alpha^2\beta^2 &\geq 0 \\
\alpha^2\beta^2 - \alpha^2\beta - \alpha\beta^2 + \alpha\beta - \alpha^2\beta^2 &\geq 0 \\
\alpha\beta(-\alpha-\beta+1) &\geq 0 \\
&\Rightarrow \quad \alpha+\beta \leq 1
\end{aligned}
$$

Therefore the function is concave iff $\alpha+\beta \leq 1$.

10.

(a)

$$
\begin{aligned}
\pi &= xp_X + yp_Y - \left(4x^2 + xy + 2y^2\right) \\
&= x(150 - 5x + y) + y(30 + 2x - 2y) - 4x^2 - xy - 2y^2 \\
&= 150x - 5x^2 + xy + 30y + 2xy - 2y^2 - 4x^2 - xy - 2y^2 \\
&= -9x^2 + 2xy + 150x - 4y^2 + 30y
\end{aligned}
$$

FOC:

$$
\begin{aligned}
\mathbf{D}\pi(x,y) = \begin{bmatrix} \partial\pi/\partial x \\ \partial\pi/\partial y \end{bmatrix} &= 0 \\
\Rightarrow \begin{bmatrix} -18x + 2y + 150 \\ 2x - 8y + 30 \end{bmatrix} &= 0
\end{aligned}
$$

Solving simultaneously:

$$
\begin{aligned}
x^* &= 9 \\
y^* &= 6
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
p_X^* &= 150 - 5(9) + 6 = 111 \\
p_Y^* &= 30 + 2(9) - 2(6) = 36
\end{aligned}
$$

And:

$$\begin{aligned}\pi_{\max} &= xp_X + yp_Y - \left(4x^2 + xy + 2y^2\right) \\ &= 9\,(111) + 6\,(36) - \left(4\,(9)^2 + 9\,(6) + 2\,(6)^2\right) \\ &= 765\end{aligned}$$

(b) SOC:

$$\begin{aligned}\mathbf{H} = \mathbf{D}^2\pi\,(x,y) &= \begin{bmatrix} \partial^2\pi/\partial x^2 & \partial\pi/\partial x\partial y \\ \partial\pi/\partial y\partial x & \partial^2\pi/\partial y^2 \end{bmatrix} \\ &= \begin{bmatrix} -18 & 2 \\ 2 & -8 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}|\mathbf{H}_1| &= |-18| < 0 \\ |\mathbf{H}_2| &= \begin{vmatrix} -18 & 2 \\ 2 & -8 \end{vmatrix} = 140 > 0\end{aligned}$$

Thus, **H** is negative definite and hence we have a maximum.

11.

$$\begin{aligned}\mathbf{D}z &= \begin{bmatrix} \partial z/\partial x \\ \partial z/\partial y \end{bmatrix} \\ &= \begin{bmatrix} -4x - 21 + 8y \\ 8x + 3y^2 - 4y + 6 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{H} = \mathbf{D}^2 z &= \begin{bmatrix} \partial^2 z/\partial x^2 & \partial^2 z/\partial x\partial y \\ \partial^2 z/\partial y\partial x & \partial^2 z/\partial y^2 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 8 \\ 8 & 6y - 4 \end{bmatrix}\end{aligned}$$

FOCs

$$-4x - 21 + 8y = 0 \qquad (1)$$

$$8x + 3y^2 - 4y + 6 = 0 \qquad (2)$$

Solve simultaneously:

From (1)

$$x = 2y - \frac{21}{4} \tag{3}$$

Substitute (3) into (2)

$$\begin{aligned} 8\left(2y - \frac{21}{4}\right) + 3y^2 - 4y + 6 &= 0 \\ \Rightarrow 3y^2 + 12y - 36 &= 0 \\ \Rightarrow y^2 + 4y - 12 &= 0 \\ \Rightarrow (y+6)(y-2) &= 0 \\ \Rightarrow y &= -6, 2 \end{aligned}$$

Therefore, candidates for extrema are $x = -\frac{69}{4}, y = -6, z = \frac{2169}{8}$ and $x = -\frac{5}{4}, y = 2, z = \frac{121}{8}$.

SOCs:

The definiteness of the Hessian matrix depends on the value of $y$, so the extrema cannot be absolute (global) but only relative (local) extrema.

We must determine the definiteness of the Hessian matrix at the candidate extrema:

At $x = -\frac{69}{4}, y = -6$ :

$$\begin{aligned} \mathbf{H} &= \begin{bmatrix} -4 & 8 \\ 8 & -40 \end{bmatrix} \\ |\mathbf{H}_1| &= |-4| = -4 < 0 \\ |\mathbf{H}_2| &= \begin{vmatrix} -4 & 8 \\ 8 & -40 \end{vmatrix} = 96 > 0 \end{aligned}$$

At this point, $\mathbf{H}$ is negative definite. So this point is a *local maximum.*

At $x = -\frac{5}{4}, y = 2$ :

$$\begin{aligned} \mathbf{H} &= \begin{bmatrix} -4 & 8 \\ 8 & 8 \end{bmatrix} \\ |\mathbf{H}_1| &= |-4| = -4 < 0 \\ |\mathbf{H}_2| &= \begin{vmatrix} -4 & 8 \\ 8 & 8 \end{vmatrix} = -96 < 0 \end{aligned}$$

At this point, $\mathbf{H}$ is neither positive nor negative definite. So this point is neither a local minimum nor a local maximum.

12.

(a)

$$\max_{x_1,x_2} \mathcal{L} = x_1^{3/4} x_2^{1/4} + \lambda \left(m - p_1 x_1 - p_2 x_2\right)$$

FOCs:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{3}{4} x_1^{-1/4} x_2^{1/4} - \lambda p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= \frac{1}{4} x_1^{3/4} x_2^{-3/4} - \lambda p_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= m - p_1 x_1 - p_2 x_2 = 0
\end{aligned}$$

Solving simultaneously:

$$\begin{aligned}
x_1^* &= \frac{3m}{4p_1} \\
x_2^* &= \frac{m}{4p_2}
\end{aligned}$$

(b) SOC:

$$\mathbf{BH} = \begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & -\frac{3}{16} x_1^{-5/4} x_2^{1/4} & \frac{3}{16} x_1^{-1/4} x_2^{-3/4} \\ -p_2 & \frac{3}{16} x_1^{-1/4} x_2^{-3/4} & -\frac{3}{16} x_1^{3/4} x_2^{-7/4} \end{bmatrix}$$

$$\begin{aligned}
\left|\mathbf{BH}_2\right| &= \begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & -\frac{3}{16} x_1^{-5/4} x_2^{1/4} & \frac{3}{16} x_1^{-1/4} x_2^{-3/4} \\ -p_2 & \frac{3}{16} x_1^{-1/4} x_2^{-3/4} & -\frac{3}{16} x_1^{3/4} x_2^{-7/4} \end{vmatrix} \\
&= \frac{1}{16} x_1^{-5/4} x_2^{-7/4} \left(3p_1^2 x_1^2 + 6 p_1 p_2 x_1 x_2 + 3 p_2^2 x_2^2\right) \\
&> 0
\end{aligned}$$

Therefore bordered Hessian is negative definite, and we have a constrained maximum.

(c) It is the marginal utility of money.

13.

(a) Constant returns to scale.

$9(\lambda K)^{1/3}(\lambda L)^{2/3} = \lambda 9K^{1/3}L^{2/3} \Rightarrow$ constant returns to scale.

OR Sum the exponents $\frac{1}{3} + \frac{2}{3} = 1 \Rightarrow$ constant returns to scale.

(b)

$$\begin{aligned} f_K &= 3K^{-2/3}L^{2/3} \\ f_{KK} &= -2K^{-5/3}L^{2/3} < 0 \Rightarrow \text{ diminishing returns to capital} \end{aligned}$$

$$\begin{aligned} f_L &= 6K^{1/3}L^{-1/3} \\ f_{LL} &= -2K^{1/3}L^{-4/3} < 0 \Rightarrow \text{ diminishing returns to labour} \end{aligned}$$

(c) Problem: $\min\limits_{K,L} 4K + 16L$ subject to $\overline{Q} = 9K^{1/3}L^{2/3}$

Lagrangian:

$$\mathcal{L} = 4K + 16L + \lambda\left(\overline{Q} - 9K^{1/3}L^{2/3}\right)$$

FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial K} &= 4 - \lambda 3K^{-2/3}L^{2/3} = 0 && (1) \\ \frac{\partial \mathcal{L}}{\partial L} &= 16 - \lambda 6K^{1/3}L^{-1/3} = 0 && (2) \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= \overline{Q} - 9K^{1/3}L^{2/3} = 0 && (3) \end{aligned}$$

(1) ÷ (2)

$$\begin{aligned} \frac{1}{4} &= \frac{1L}{2K} \\ \Rightarrow L &= \frac{1}{2}K && (4) \end{aligned}$$

Sub (4) into (3)

$$\begin{aligned} \overline{Q} - 9K^{1/3}\left(\frac{1}{2}K\right)^{2/3} &= 0 \\ \frac{9}{2^{2/3}}K &= \overline{Q} \\ K^* &= \frac{2^{2/3}}{9}\overline{Q} \\ &= 0.176\overline{Q} \end{aligned}$$

$$
\begin{aligned}
L^* &= \frac{1}{2}\left(\frac{2^{2/3}}{9}\overline{Q}\right) \\
&= \frac{1}{9\left(2^{1/3}\right)}\overline{Q} \\
&= 0.088\overline{Q}
\end{aligned}
$$

$$
\begin{aligned}
4-\lambda 3K^{-2/3}L^{2/3} &= 0 \\
\Rightarrow \lambda^* &= \frac{4}{3}K^{2/3}L^{-2/3} \\
&= \frac{4}{3}\left(\frac{2^{2/3}}{9}\overline{Q}\right)^{2/3}\left(\frac{1}{9\left(2^{1/3}\right)}\overline{Q}\right)^{-2/3} \\
&= \frac{4}{3}\left(\frac{2^{4/9}}{9^{2/3}}\right)\left(9^{2/3}2^{2/9}\right) \\
&= \frac{4}{3}\left(2^{7/9}\right) \\
&= \frac{2^{25/9}}{3} \\
&= 2.286
\end{aligned}
$$

(d) SOC:

$$
\overline{\mathbf{H}} = \begin{bmatrix} 0 & 3K^{-2/3}L^{2/3} & 6K^{1/3}L^{-1/3} \\ 3K^{-2/3}L^{2/3} & 2\lambda K^{-5/3}L^{2/3} & -2\lambda K^{-2/3}L^{-1/3} \\ 6K^{1/3}L^{-1/3} & -2\lambda K^{-2/3}L^{-1/3} & 2\lambda K^{1/3}L^{-4/3} \end{bmatrix}
$$

$$
\begin{aligned}
\left|\overline{\mathbf{H}}_2\right| &= \left|\overline{\mathbf{H}}\right| = 0 - 3K^{-2/3}L^{2/3}\left(6\lambda K^{-1/3}L^{-2/3} + 12\lambda K^{-1/3}L^{-2/3}\right) \\
&\quad +6K^{1/3}L^{-1/3}\left(-6\lambda K^{-4/3}L^{1/3} - 12\lambda K^{-4/3}L^{1/3}\right) \\
&= -3K^{-2/3}L^{2/3}\left(18\lambda K^{-1/3}L^{-2/3}\right) + 6K^{1/3}L^{-1/3}\left(-18\lambda K^{-4/3}L^{1/3}\right) \\
&= -54\lambda K^{-1} - 108\lambda K^{-1} \\
&= -162\lambda K^{-1} \\
&= -\frac{162\lambda}{K}
\end{aligned}
$$

At $K^*, \lambda^*$

$$\left|\overline{\mathbf{H}}_2\right| = -\frac{162\lambda}{K} < 0$$

Therefore, the point is a constrained minimum.

(e) It is the marginal cost.

14. Rewrite the constraints so they are in the form $g_i(\mathbf{x}) \leq 0$ :

$$\begin{aligned} x + 2y - 2 &\leq 0 \\ 2x - y + 1 &\leq 0 \end{aligned}$$

(a) Set up the Lagrangian

$$\mathcal{L} = -4x^2 - 2xy - 4y^2 - \lambda\left(x + 2y - 2\right) - \mu\left(2x - y + 1\right)$$

The Kuhn-Tucker conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= -8x - 2y - \lambda - 2\mu = 0 \quad (1) \\ \frac{\partial \mathcal{L}}{\partial y} &= -2x - 8y - 2\lambda + \mu = 0 \quad (2) \\ \lambda &\geq 0, x + 2y \leq 2 \text{ and } \lambda\left(x + 2y - 2\right) = 0 \\ \mu &\geq 0, 2x - y \leq -1 \text{ and } \mu\left(2x - y + 1\right) = 0 \end{aligned}$$

Consider the possible cases in turn:

(a) **Case 1**: $\lambda > 0, \mu > 0$
If $\lambda > 0$, then complementary slackness implies $x + 2y - 2 = 0$.
If $\mu > 0$, then complementary slackness implies $2x - y + 1 = 0$.
Solving these equations simultaneously gives $x = 0, y = 1$.
Substituting $x = 0, y = 1$ into equations (1) and (2) gives

$$\begin{aligned} -2 - \lambda - 2\mu &= 0 \\ -8 - 2\lambda + \mu &= 0 \end{aligned}$$

Solving simultaneously gives $\lambda = -\frac{18}{5}, \mu = \frac{4}{5}$ which violates $\lambda > 0$.
**Case 2**: $\lambda > 0, \mu = 0$
If $\lambda > 0$, then complementary slackness implies $x + 2y - 2 = 0$.
Substituting $\mu = 0$ into equations (1) and (2) gives

$$\begin{aligned} -8x - 2y - \lambda &= 0 \\ -2x - 8y - 2\lambda &= 0 \end{aligned}$$

Solving these three equations simultaneously gives $x = \frac{1}{4}, y = \frac{7}{8}, \lambda = -\frac{15}{4}$ which violates $\lambda > 0$.

**Case 3**: $\lambda = 0, \mu > 0$

If $\mu > 0$, then complementary slackness implies $2x - y + 1 = 0$.

Substituting $\lambda = 0$ into equations (1) and (2) gives

$$\begin{aligned} -8x - 2y - 2\mu &= 0 \\ -2x - 8y + \mu &= 0 \end{aligned}$$

Solving these three equations simultaneously gives $x = -\frac{3}{8}, y = \frac{1}{4}, \mu = \frac{5}{4}$ which satisfies all conditions.

**Case 4**: $\lambda = 0, \mu = 0$

Substituting $\lambda = 0, \mu = 0$ into equations (1) and (2) gives

$$\begin{aligned} -8x - 2y &= 0 \\ -2x - 8y &= 0 \end{aligned}$$

which is satisfied if and only if $x = 0, y = 0$ which violates $2x - y + 1 \leq 0$. So no solution candidates here.

**Therefore:**

The Kuhn Tucker necessary conditions for a maximum are satisfied under Case 3 where $x = -\frac{3}{8}, y = \frac{1}{4}, \lambda = 0, \mu = \frac{5}{4}$.

(This point also satisfies the sufficient conditions for a maximum and so is the optimum point.)

15. Two possible Lagrangians and associated FOCs:

(a)

$$\mathcal{L} = 3.6x - 0.4x^2 + 1.6y - 0.2y^2 + \lambda(10 - 2x - y)$$

FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= 3.6 - 0.8x - 2\lambda \leq 0, \quad x \geq 0, \quad (x)(3.6 - 0.8x - 2\lambda) = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= 1.6 - 0.4y - \lambda \leq 0, \quad y \geq 0, \quad (y)(1.6 - 0.4y - \lambda) = 0 \\ \lambda &\geq 0, \quad 2x + y \leq 10, \quad (\lambda)(2x + y - 10) = 0 \end{aligned}$$

**OR**

(b)

$$\mathcal{L} = 3.6x - 0.4x^2 + 1.6y - 0.2y^2 + \lambda_1 (10 - 2x - y) + \lambda_2 (x) + \lambda_3 (y)$$

FOCs:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 3.6 - 0.8x - 2\lambda_1 + \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= 1.6 - 0.4y - \lambda_1 + \lambda_3 = 0 \\
\lambda_1 &\geq 0, \quad 2x + y \leq 10, \quad (\lambda_1)(2x + y - 10) = 0 \\
\lambda_2 &\geq 0, \quad x \geq 0, \quad (\lambda_2)(x) = 0 \\
\lambda_3 &\geq 0, \quad y \geq 0, \quad (\lambda_3)(y) = 0
\end{aligned}$$

Consider the various cases (Working now with the Lagrangian and FOCs from (a)):

$$\begin{aligned}
(\text{I}) &: 2x + y = 10, x = 0, y = 0 \\
(\text{II}) &: 2x + y < 10, x = 0, y = 0 \\
(\text{III}) &: 2x + y = 10, x > 0, y = 0 \\
(\text{IV}) &: 2x + y = 10, x = 0, y > 0 \\
(\text{V}) &: 2x + y < 10, x > 0, y = 0 \\
(\text{VI}) &: 2x + y < 10, x = 0, y > 0 \\
(\text{VII}) &: 2x + y = 10, x > 0, y > 0 \\
(\text{VIII}) &: 2x + y < 10, x > 0, y > 0
\end{aligned}$$

**Case (I)**: $2x + y = 10, x = 0, y = 0$

Inconsistent, since $x = 0, y = 0$ violates $2x + y = 10$

**Case (II)**: $2x + y < 10, x = 0, y = 0$

$$\begin{aligned}
2x + y &< 10 \Rightarrow \lambda = 0 \text{ (from complementary slackness condition)} \\
\text{Substituting } \lambda &= 0, x = 0, y = 0 \text{ into:} \\
\frac{\partial \mathcal{L}}{\partial x} &= 3.6 \not\leq 0 \text{ and } \frac{\partial \mathcal{L}}{\partial y} = 1.6 \not\leq 0 \\
&\phantom{=} \text{So no solution here.}
\end{aligned}$$

**Case (III)**: $2x + y = 10, x > 0, y = 0$

$$2x + y = 10, y = 0 \Rightarrow x = 5$$

$$
\begin{aligned}
x > 0 \Rightarrow 3.6 - 0.8x - 2\lambda &= 0 \text{ (from complementary slackness condition)} \\
\text{Sub in } x = 5: \; 3.6 - 0.8\,(5) - 2\lambda &= 0 \\
\Rightarrow \lambda &= -0.2 \\
\text{violates } \lambda &\geq 0
\end{aligned}
$$

**Case (IV)**: $2x + y = 10, x = 0, y > 0$

$$2x + y = 10, x = 0 \Rightarrow y = 10$$

$$
\begin{aligned}
y > 0 \Rightarrow 1.6 - 0.4y - \lambda &= 0 \text{ (from complementary slackness condition)} \\
\text{Sub in } y = 10: \; 1.6 - 0.4\,(10) - \lambda &= 0 \\
\Rightarrow \lambda &= -2.4 \\
\text{violates } \lambda &\geq 0
\end{aligned}
$$

**Case (V)**: $2x + y < 10, x > 0, y = 0$

$$2x + y < 10 \Rightarrow \lambda = 0 \text{ (from complementary slackness condition)}$$

$$
\begin{aligned}
\text{Sub in } \lambda &= 0, y = 0 \text{ into} \\
\frac{\partial \mathcal{L}}{\partial y} &= 1.6 - 0.4y - \lambda = 1.6 \nleq 0 \\
&\text{So no solution here}
\end{aligned}
$$

**Case (VI)**: $2x + y < 10, x = 0, y > 0$

$$2x + y < 10 \Rightarrow \lambda = 0 \text{ (from complementary slackness condition)}$$

$$
\begin{aligned}
\text{Sub in } \lambda &= 0, x = 0 \text{ into} \\
\frac{\partial \mathcal{L}}{\partial x} &= 3.6 - 0.8x - 2\lambda = 3.6 \nleq 0 \\
&\text{So no solution here}
\end{aligned}
$$

**Case (VII)**: $2x + y = 10, x > 0, y > 0$

$$
\begin{aligned}
x > 0 \Rightarrow 3.6 - 0.8x - 2\lambda &= 0 \text{ (from complementary slackness condition)} \\
y > 0 \Rightarrow 1.6 - 0.4y - \lambda &= 0 \text{ (from complementary slackness condition)} \\
2x + y &= 10
\end{aligned}
$$

This gives the augmented matrix:

$$\left[\begin{array}{ccc|c} -0.8 & 0 & -2 & -3.6 \\ 0 & -0.4 & -1 & -1.6 \\ 2 & 1 & 0 & 10 \end{array}\right]$$

Solving yields:

$$\begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} = \begin{bmatrix} 3.5 \\ 3 \\ 0.4 \end{bmatrix}$$

This satisfies all conditions and is a possible solution.

**Case (VIII)**: $2x + y < 10, x > 0, y > 0$

$$\begin{aligned} x > 0 &\Rightarrow 3.6 - 0.8x - 2\lambda = 0 \text{ (from complementary slackness condition)} \\ y > 0 &\Rightarrow 1.6 - 0.4y - \lambda = 0 \text{ (from complementary slackness condition)} \\ 2x + y < 10 &\Rightarrow \lambda = 0 \text{ (from complementary slackness condition)} \end{aligned}$$

Solving:

$$\begin{aligned} 3.6 - 0.8x - 2(0) &= 0 \Rightarrow x = 4.5 \\ 1.6 - 0.4y - 0 &= 0 \Rightarrow y = 4 \end{aligned}$$

But then:

$$\begin{aligned} 2x + y &= 2(4.5) + 4 = 13 \nleq 10 \\ &\text{So no solution here.} \end{aligned}$$

Therefore, the only solution is $\begin{bmatrix} x \\ y \\ \lambda \end{bmatrix}$

16.

(a)

$$\mathcal{L} = u(x_1, x_2) - \lambda(p_1x_1 + p_2x_2 - m) - \mu(t_1x_1 + t_2x_2 - T)$$

The solution that provides the maximum value must satisfy:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= u_1 - \lambda p_1 - \mu t_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= u_2 - \lambda p_2 - \mu t_2 = 0 \\ \lambda &\geq 0, p_1x_1 + p_2x_2 \leq m \text{ and } \lambda(p_1x_1 + p_2x_2 - m) = 0 \\ \mu &\geq 0, t_1x_1 + t_2x_2 \leq T \text{ and } \mu(t_1x_1 + t_2x_2 - T) = 0 \end{aligned}$$

(b) It will not be the case that $\lambda = 0$ and $\mu = 0$ because complementary slackness implies that $p_1x_1 + p_2x_2 < m$ and $t_1x_1 + t_2x_2 < T$ and then more of one or both of the goods can be consumed, which would raise utility.

It is also unlikely that both $\lambda > 0$ and $\mu > 0$ since this would mean that both constraints would be binding, i.e. $p_1x_1 + p_2x_2 = m$ and $t_1x_1 + t_2x_2 = T$.

It is more likely that either $\lambda > 0$ and $\mu = 0$, in which case the budget constraint is binding, or $\lambda = 0$ and $\mu > 0$ in which case the time constraint is binding.