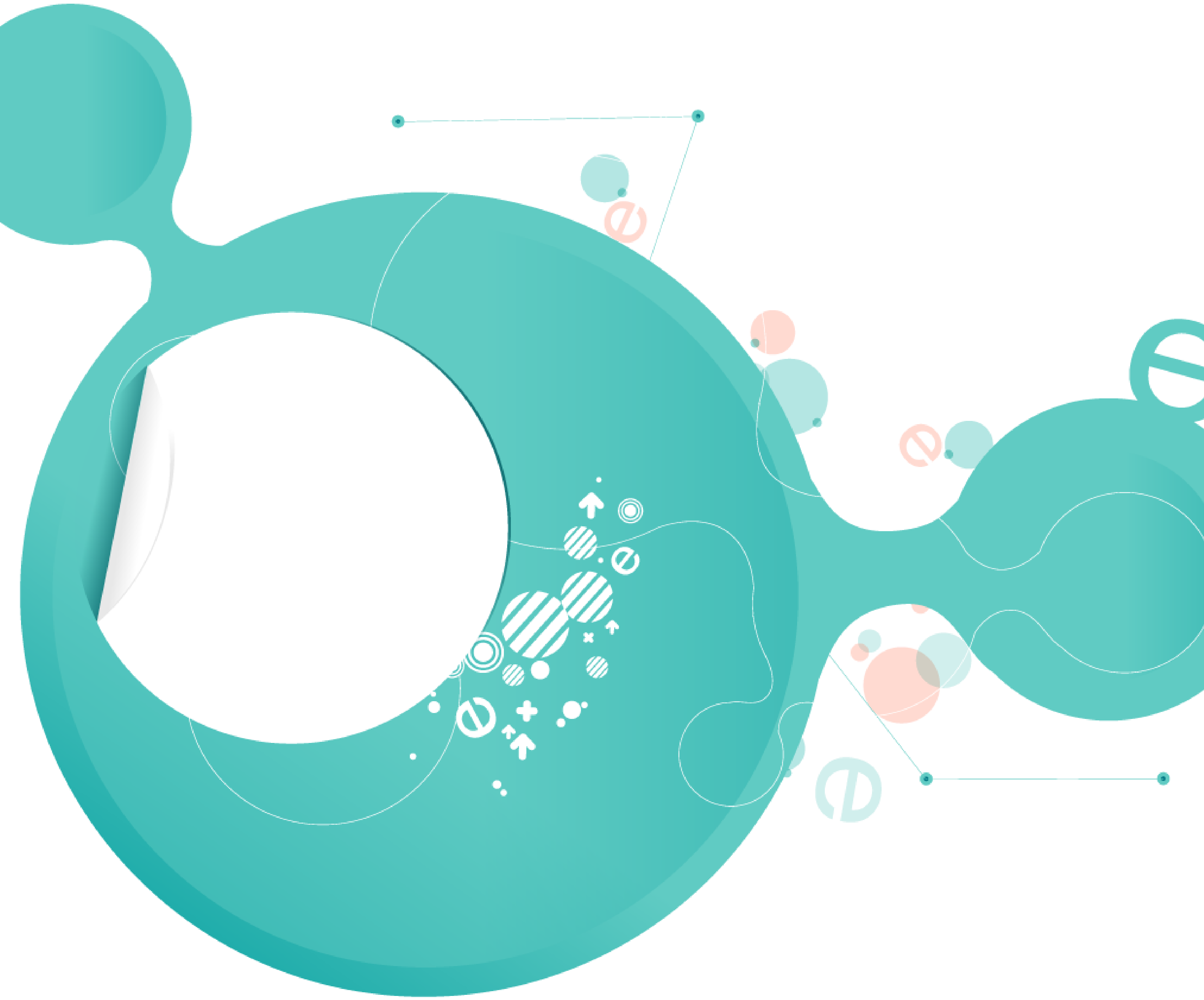
Mathematics for Economists

Tutorial Questions – Linear Algebra



Tutorial 1: Linear Algebra

ECO4112F 2011

1. Suppose

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2q \\ 6 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} p+2 \\ -5 \\ 3r \end{bmatrix}$$

   If $x = 2y$, find p, q, r.

2. Which of the following sets of vectors are linearly dependent?

   (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

   (d) $\begin{bmatrix} 13 \\ 7 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \\ 8 \end{bmatrix}$

3. Let

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & -1 \\ -6 & 3 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 5 & -3 & 2 \\ 3 & -8 & 2 \end{bmatrix}$$

   Find

   (a) $\mathbf{A} + \mathbf{B}$

   (b) $6\mathbf{A}$

   (c) $-3\mathbf{B}$

   (d) $4\mathbf{A} - 5\mathbf{B}$

   (e) $\mathbf{AB}$

4. Let

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 3 & -2 \\ 4 & 2 \\ 5 & -3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 & 1 & 2 \\ -1 & 0 & -2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 3 & -1 & 4 \\ -4 & 2 & 1 & 0 \end{bmatrix}$$

Find

(a) **AB**
(b) **BA**
(c) **AC**
(d) **CA**

5. Solve the system $\mathbf{Ax} = \mathbf{b}$ in each of the following cases (Hint: note that these are echelon matrices):

(a) $\mathbf{A} = \begin{bmatrix} 3 & -5 & 0 & 1 \\ 0 & 2 & -1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 7 & 3 & -2 & 1 \\ 0 & 8 & 4 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -2 \\ 0 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 6 & 3 & 9 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -8 \\ 2 \end{bmatrix}$

(d) $\mathbf{A} = \begin{bmatrix} 7 & 3 & -2 & 1 \\ 0 & 8 & 4 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -2 \\ -1 \end{bmatrix}$

(e) $\mathbf{A} = \begin{bmatrix} 6 & 3 & 9 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -8 \\ 0 \end{bmatrix}$

6. Consider the following augmented matrix, where k is a constant

$$\mathbf{A} = \left[\begin{array}{ccc|c} 7 & 2 & -1 & 2 \\ 2 & -1 & 1 & 6 \\ 7 & 7 & 0 & 0 \\ -5 & -8 & 1 & k \end{array}\right]$$

   (a) Find the row echelon form of $\mathbf{A}$.
   (b) Determine for which value of k this system of equations will have a unique solution. Discuss the rank of the coefficient and augmented matrices under this value of k.
   (c) Determine for which value of k this system of equations will have no solution. Discuss the rank of the coefficient and augmented matrices under this value of k.
   (d) Using the value of k from (b) solve for the values of x_1, x_2 and x_3.

7. Use Gaussian elimination to solve the system $\mathbf{Ax} = \mathbf{b}$ in each of the following cases, and discuss the ranks of $\mathbf{A}$ and the augmented matrix.

   (a) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 4 & 1 \\ 4 & 3 & 3 & 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
   (b) $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$
   (c) $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

8. Use Gaussian elimination to invert the matrices

$$\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 5 \\ 1 & 7 & 5 \\ 5 & 10 & 15 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 4 \\ 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix}$$

   Hence solve the systems of equations:

   (a) $3x_1 - x_2 = 5, \;\; 2x_1 + x_2 = 1$
   (b) $3x_2 + 4x_3 = 5, \;\; 2x_1 + 3x_3 = -1, \;\; -x_1 + x_2 = 2$

9. Find the following determinants:

   (a) $\begin{vmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{vmatrix}$

   (b) $\begin{vmatrix} 4 & 0 & 2 \\ 6 & 0 & 3 \\ 8 & 2 & 3 \end{vmatrix}$

   (c) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

   (d) $\begin{vmatrix} 1 & 0 & 3 & 2 \\ 4 & -1 & 0 & 1 \\ 2 & 1 & 0 & 3 \\ -1 & 2 & 3 & -1 \end{vmatrix}$

   (e) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix}$

10. Find the inverses of the following matrices, using the ‘adj-over-det’ formula:

    (a) $\begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$

    (b) $\begin{bmatrix} 1 & 0 \\ 9 & 2 \end{bmatrix}$

    (c) $\begin{bmatrix} 4 & -2 & 1 \\ 7 & 3 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

    (d) $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{bmatrix}$

11. Find the inverses of the following matrices, using either Gaussian elimination or the ‘adj-over-det’ formula:

    (a) $\begin{bmatrix} 7 & 7 \\ 3 & -1 \end{bmatrix}$

    (b) $\begin{bmatrix} 7 & 6 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 5 & 0 \\ 2 & 1 & 6 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 & 9 \\ 1 & 0 & 0 \\ 7 & 8 & 4 \end{bmatrix}$

12. Determine the definiteness of the symmetric matrices

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

13. Determine the values of t for which the symmetric matrix $\begin{bmatrix} 2t & 2 \\ 2 & t \end{bmatrix}$ is

(a) positive definite;
(b) positive semidefinite but not positive definite;
(c) negative definite;
(d) negative semidefinite but not negative definite;
(e) indefinite.

14. Solve the following systems of equations using Gaussian elimination:

(a) $5x + 3y = 30$
$6x - 2y = 8$

(b) $7x - y - z = 0$
$z + 10x - 2y = 8$
$6x + 3y - 2z = 7$

15. Solve the national income system below using the inverse method:

$$\begin{aligned} Y &= C + I_0 + G_0 \\ C &= a + bY \end{aligned}$$

Solve for C and Y.

16. Solve the national income system below using the inverse method:

$$\begin{aligned} Y &= C + I_0 + G_0 \\ C &= a + b(Y - T) \qquad (a > 0, 0 < b < 1, d > 0, 0 < t < 1) \\ T &= d + tY \end{aligned}$$

Solve for Y, C and T.

17. Solve the market system below for equilibrium price and quantity, where Q_d, Q_s and P are endogenous, and G and N are exogenous.

$$\begin{aligned} Q_d &= \alpha - \beta P + \gamma G \\ Q_s &= \theta + \lambda P - \phi N \\ Q_d &= Q_s \end{aligned}$$

18. Use Cramer's rule to solve the following equation systems:

    (a) $3x - 2y = 11$
    $2x + y = 12$

    (b) $8x - y = 15$
    $y + 5z = 1$
    $2x + 3z = 4$

19. Use Cramer's rule to solve for Y, C and G in the national income system below, and state what restriction is required on the parameters for a solution to exist. Give the economic meaning of the parameter g. Also, find the tax multiplier and investment multiplier, and give the economic intuition behind their signs. Show also that an increase in tax has a negative impact on consumption.

$$\begin{aligned} Y &= C + I_0 + G \\ C &= a + b(Y - T_0) \qquad (a > 0, 0 < b < 1, 0 < g < 1) \\ G &= gY \end{aligned}$$

20. Consider the following simple Keynesian macroeconomic model:

$$\begin{aligned} Y &= C + I + G \\ C &= 200 + 0.8Y \\ I &= 1000 - 2000R \end{aligned}$$

where the endogenous variables are Y, C and I, and the exogenous variables are G and R. Using Cramer's rule, solve for Y, and evaluate the effect of a R50 billion decrease in government spending on national income.

21. Use Cramer's Rule to solve for the equilibrium price and quantity of BMWs in the automobile market system below, where G is the price of substitute goods (e.g. Mercedes Benz), and N is the price of inputs. Show that an increase in the price of inputs will reduce equilibrium quantity and increase equilibrium price, while a decrease in the price of substitutes will lead to a decrease in equilibrium price and quantity of BMWs. Be sure to explain the economic reasoning in your argument.

$$\begin{aligned} Q_d &= a - bP + \phi G \\ Q_s &= -c + dP - \lambda N \qquad (a, b, \phi, c, d, \lambda > 0) \\ Q_d &= Q_s \end{aligned}$$

22. Consider the system of equations

$$\begin{aligned} 3x_1 + 2x_2 + 5x_3 &= 8 \\ 2x_1 + x_2 + 2x_3 &= 7 \\ 8x_1 + 5x_2 + 12x_3 &= 23 \end{aligned}$$

Discuss the independence and consistency of these equations. Derive the solution.

Tutorial 1: Linear Algebra

SELECTED SOLUTIONS

ECO4112F 2011

1. $p = -\frac{1}{2}, q = -5, r = 1.$

2. (a), (c) and (d) are linearly dependent, (b) is linearly independent.

3. (a) $\mathbf{A}+\mathbf{B} = \begin{bmatrix} 9 & -1 & 1 \\ -3 & -5 & 2 \end{bmatrix}$

   (b) $6\mathbf{A} = \begin{bmatrix} 24 & 12 & -6 \\ -36 & 18 & 0 \end{bmatrix}$

   (c) $-3\mathbf{B} = \begin{bmatrix} -15 & 9 & -6 \\ -9 & -24 & -6 \end{bmatrix}$

   (d) $4\mathbf{A}-5\mathbf{B} = \begin{bmatrix} 16 & 8 & -4 \\ -24 & 12 & 0 \end{bmatrix} + \begin{bmatrix} -25 & 15 & -10 \\ -15 & 40 & -10 \end{bmatrix} = \begin{bmatrix} -9 & 23 & -14 \\ -39 & 52 & -10 \end{bmatrix}$

   (e) **AB** Matrices not conformable.

4. (a)

$$\mathbf{AB} = \begin{bmatrix} -4 & -1 & -2 \\ 14 & 3 & 10 \\ 14 & 4 & 4 \\ 23 & 5 & 16 \end{bmatrix}$$

   (b) **BA** is not defined because the matrices are not conformable.

   (c)

$$\mathbf{AC} = \begin{bmatrix} -1 & -3 & 1 & -4 \\ 11 & 5 & -5 & 12 \\ -4 & 16 & -2 & 16 \\ 17 & 9 & -15 & 20 \end{bmatrix}$$

   (d)

$$\mathbf{CA} = \begin{bmatrix} 24 & -20 \\ 14 & -2 \end{bmatrix}$$

5. (a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} \frac{5}{6} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -5 \\ -2 \end{bmatrix}$

(c) No solution (The last equation $0 = 2$ is inconsistent)

(d) No solution (The last equation $0 = -1$ is inconsistent)

(e) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ 0 \\ 0 \\ -4 \end{bmatrix} + \lambda \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}$

6. (a) Row echelon form (no fractions):

$$\left[\begin{array}{ccc|c} 7 & 2 & -1 & 2 \\ 0 & -11 & 9 & 38 \\ 0 & 0 & -56 & -168 \\ 0 & 0 & 0 & -56k + 336 \end{array}\right]$$

Row echelon form (with fractions):

$$\left[\begin{array}{ccc|c} 7 & 2 & -1 & 2 \\ 0 & 5 & 1 & -2 \\ 0 & 0 & \frac{8}{5} & \frac{24}{5} \\ 0 & 0 & 0 & k - 6 \end{array}\right]$$

(b) $k = 6$. Rank of coefficient matrix $= 3 =$ Rank of augmented matrix.

(c) $k \neq 6$. Rank of coefficient matrix $= 3 < 4 =$ Rank of augmented matrix.

(d) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

7. (a)

$$\begin{bmatrix} 2 & -1 & 4 & 1 & | & 0 \\ 4 & 3 & 3 & 7 & | & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & -1 & 4 & 1 & | & 0 \\ 0 & 5 & -5 & 5 & | & -1 \end{bmatrix} \quad (2) - 2 \times (1)$$

Columns 1 and 2 are basic, so we assign arbitrary values to the non-basic columns, $x_3 = \lambda, x_4 = \mu$. Then

$$\begin{aligned} 5x_2 - 5\lambda + 5\mu &= -1 \\ \Rightarrow x_2 &= -\frac{1}{5} + \lambda - \mu \end{aligned}$$

$$\begin{aligned} 2x_1 - x_2 + 4\lambda + \mu &= 0 \\ \Rightarrow x_1 &= \frac{1}{2}\left(-\frac{1}{5} + \lambda - \mu - 4\lambda - \mu\right) \\ &= \frac{1}{2}\left(-\frac{1}{5} - 3\lambda - 2\mu\right) \\ &= -\frac{1}{10} - \frac{3}{2}\lambda - \mu \end{aligned}$$

The complete solution in vector form is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} \\ -\frac{1}{5} \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The rank of $\mathbf{A}$ = the rank of the augmented matrix = 2.

(b)

$$\begin{bmatrix} 2 & 3 & 4 & | & 1 \\ 5 & 6 & 7 & | & 2 \\ 8 & 9 & 10 & | & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 3 & 4 & | & 1 \\ 0 & -\frac{3}{2} & -3 & | & -\frac{1}{2} \\ 0 & -3 & -6 & | & 0 \end{bmatrix} \quad \begin{array}{l} (2) - \frac{5}{2} \times (1) \\ (3) - 4 \times (1) \end{array}$$

$$\rightarrow \begin{bmatrix} 2 & 3 & 4 & | & 1 \\ 0 & -\frac{3}{2} & -3 & | & -\frac{1}{2} \\ 0 & 0 & 0 & | & 1 \end{bmatrix} \quad (3) - 2 \times (2)$$

There is no solution. The rank of $\mathbf{A} = \mathbf{2} < \mathbf{3}$ = rank of the augmented matrix.

(c)

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 2 \\ 8 & 9 & 10 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -\frac{3}{2} & -3 & -\frac{1}{2} \\ 0 & -3 & -6 & -1 \end{array}\right] \quad \begin{array}{l} (2) - \frac{5}{2} \times (1) \\ (3) - 4 \times (1) \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -\frac{3}{2} & -3 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (3) - 2 \times (2)$$

We can ignore the last row of zeros and solve the system above it. The first and second columns are basic, so we let $x_3 = \lambda$. Then

$$\begin{aligned} -\frac{3}{2}x_2 - 3\lambda &= -\frac{1}{2} \\ \Rightarrow x_2 &= -\frac{2}{3}\left(-\frac{1}{2} + 3\lambda\right) \\ &= \frac{1}{3} - 2\lambda \end{aligned}$$

$$\begin{aligned} 2x_1 + 3x_2 + 4\lambda &= 1 \\ \Rightarrow x_1 &= \frac{1}{2}\left(1 - 3\left(\frac{1}{3} - 2\lambda\right) - 4\lambda\right) \\ &= \frac{1}{2}(1 - 1 + 2\lambda) \\ &= \lambda \end{aligned}$$

The complete solution in vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The rank of $\mathbf{A}$ = rank of the augmented matrix = 2.

8. Let $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 7 & 5 \\ 5 & 10 & 15 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 3 & 4 \\ 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix}$.

To find $\mathbf{A}^{-1}$:

$$\begin{bmatrix} 3 & -1 & | & 1 & 0 \\ 2 & 1 & | & 0 & 1 \end{bmatrix}$$

$$\vdots$$

$$\rightarrow \begin{bmatrix} 1 & 0 & | & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & | & -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

To find $\mathbf{B}^{-1}$:

$$\begin{bmatrix} 1 & 3 & 5 & | & 1 & 0 & 0 \\ 1 & 7 & 5 & | & 0 & 1 & 0 \\ 5 & 10 & 15 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\vdots$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & | & -\frac{11}{8} & -\frac{1}{8} & \frac{1}{2} \\ 0 & 1 & 0 & | & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & | & \frac{5}{8} & -\frac{1}{8} & -\frac{1}{10} \end{bmatrix}$$

$$\therefore \quad \mathbf{B}^{-1} = \begin{bmatrix} -\frac{11}{8} & -\frac{1}{8} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ \frac{5}{8} & -\frac{1}{8} & -\frac{1}{10} \end{bmatrix}$$

To find $\mathbf{C}^{-1}$:

$$\begin{bmatrix} 0 & 3 & 4 & | & 1 & 0 & 0 \\ 2 & 0 & 3 & | & 0 & 1 & 0 \\ -1 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\vdots$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 3 & -4 & -9 \\ 0 & 1 & 0 & | & 3 & -4 & -8 \\ 0 & 0 & 1 & | & -2 & 3 & 6 \end{bmatrix}$$

$$\therefore \quad \mathbf{C}^{-1} = \begin{bmatrix} 3 & -4 & -9 \\ 3 & -4 & -8 \\ -2 & 3 & 6 \end{bmatrix}$$

(a) In matrix form

$$\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The coefficient matrix is $\mathbf{A}$ and we have calculated $\mathbf{A}^{-1}$. So

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \mathbf{A}^{-1} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 \\ 15 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 7 \end{bmatrix} \end{aligned}$$

(b) In matrix form

$$\begin{bmatrix} 0 & 3 & 4 \\ 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$$

The coefficient matrix is $\mathbf{C}$ and we have calculated $\mathbf{C}^{-1}$. So

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \mathbf{C}^{-1} \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -4 & -9 \\ 3 & -4 & -8 \\ -2 & 3 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \end{aligned}$$

9. (a) -6

   (b) 0

   (c) $-a^3 + 3abc - b^3 - c^3$

   (d) -90

   (e) 24 (Note this is a triangular matrix, so we can just multiply the diagonal elements to get the determinant).

10. (a) $\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 \end{bmatrix}$

    (b) $\begin{bmatrix} 1 & 0 \\ -\frac{9}{2} & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{3}{8} & \frac{1}{4} & -\frac{9}{8} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{5}{8} \\ -\frac{3}{4} & -\frac{1}{2} & \frac{13}{4} \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -\frac{1}{5} & \frac{3}{10} \\ -1 & \frac{3}{5} & \frac{1}{10} \\ 0 & \frac{2}{5} & -\frac{1}{10} \end{bmatrix}$

11. (a) $\begin{bmatrix} \frac{1}{28} & \frac{1}{4} \\ \frac{3}{28} & -\frac{1}{4} \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ 0 & \frac{1}{3} \end{bmatrix}$

(c) $\begin{bmatrix} -3 & \frac{7}{5} & 2 \\ 0 & \frac{1}{5} & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{18} & -\frac{55}{72} & \frac{1}{8} \\ \frac{1}{9} & -\frac{2}{9} & 0 \end{bmatrix}$

12.

(a) Positive definite

(b) Positive semidefinite

13.

(a) $t > \sqrt{2}$

(b) $t = \sqrt{2}$

(c) $t < -\sqrt{2}$

(d) $t = -\sqrt{2}$

(e) $-\sqrt{2} < t < \sqrt{2}$

14.

(a) $x = 3, y = 5.$

(b) $x = 1, y = 3, z = 4.$

15.

$$\begin{aligned} Y^* &= \frac{I_0 + G_0 + a}{1 - b} \\ C^* &= \frac{a + b\,(I_0 + G_0)}{1 - b} \end{aligned}$$

16.

$$\begin{aligned} Y^* &= \frac{-bd + I_0 + G_0 + a}{1 - b + bt} \\ C^* &= \frac{a - bd + b\left(I_0 + G_0\right)(1-t)}{1 - b + bt} \\ T^* &= \frac{d(1-b) + t(a + I_0 + G_0)}{1 - b + bt} \end{aligned}$$

17.

$$\begin{aligned} Q_d^* &= Q_s^* = \frac{\lambda\left(\alpha + \gamma G\right) + \beta\left(\theta - \phi N\right)}{\lambda + \beta} \\ P^* &= \frac{-\theta + \phi N + \alpha + \gamma G}{\lambda + \beta} \end{aligned}$$

18.

(a) $x = 5, y = 2.$

(b) $x = 2, y = 1, z = 0.$

19.

$$\begin{aligned} Y^* &= \frac{I_0 + a - bT_0}{1 - b - g} \\ C^* &= \frac{\left(a - bT_0\right)(1-g) + bI_0}{1 - b - g} \\ G^* &= \frac{g\left(a - bT_0 + I_0\right)}{1 - b - g} \end{aligned}$$

For a solution to exist, it must be that $(b+g) < 1$, otherwise Y, C, G will be negative. (which is impossible!) Thus, $0 < (1 - b - g) < 1$.

Tax multiplier:

$$\frac{\partial Y^*}{\partial T_0} = \frac{-b}{1 - b - g} < 0$$

As taxes are raised, national income falls. This is because raising taxes will decrease disposable income available for consumption, and consumption will fall. If consumption falls, Y falls. Note: even though taxes may be used to finance increased government spending, G, in this model, G itself depends on the amount of national income, which in turn, will depend on consumption. In addition, even if G were exogenous, it is highly unlikely that all taxes are channeled into government spending on a one-for-one basis – there will be some leakage in the system, which is why the decrease in consumption from higher taxes is not perfectly offset by an increase in G.

Investment multiplier:

$$\frac{\partial Y^*}{\partial I_0} = \frac{1}{1-b-g} > 0$$

From the national income identity, we know that $Y = C + I + G$. Thus, an increase in I, ceteris paribus, will cause an increase in Y.

Effect of tax on consumption:

$$\frac{\partial C^*}{\partial T_0} = \frac{-b\,(1-g)}{1-b-g} < 0$$

This reinforces the earlier point that if taxes are raised, less disposable income is available for spending, and consumption will fall.

20.

$$Y = \frac{1200 - 2000R + G}{0.2}$$

$$\frac{\partial Y}{\partial G} = \frac{1}{0.2} = 5$$

Thus the government spending multiplier is 5. If G decreases by R50 billion, the multiplier effect will be that Y will decrease by R250 billion.

21.

$$\begin{aligned} Q_d &= Q_s = \frac{d\,(a+\phi G) - b\,(c+\lambda N)}{b+d} \\ P &= \frac{c+\lambda N + a - \phi G}{b+d} \end{aligned}$$

Increase in the price of inputs:

$$\begin{aligned} \frac{\partial Q_d}{\partial N} &= \frac{\partial Q_s}{\partial N} = \frac{-\lambda b}{b+d} < 0 \\ \frac{\partial P}{\partial N} &= \frac{\lambda}{b+d} > 0 \end{aligned}$$

Therefore, if the price of inputs increases then equilibrium quantity will fall and equilibrium price will rise. This happens because higher input prices makes production more expensive, which causes the supply curve to shift to the left.

Decrease in the price of substitute goods:

$$\begin{aligned}\frac{\partial Q_d}{\partial G} &= \frac{\partial Q_s}{\partial G} = \frac{\phi d}{b+d} > 0\\ \frac{\partial P}{\partial N} &= \frac{\phi}{b+d} > 0\end{aligned}$$

As the price of substitutes decreases, our good becomes relatively more expensive, and thus less of it is bought (i.e. the demand curve shifts in to the left) causing a decrease in equilibrium quantity. Because the demand for our product decreases when substitute goods become relatively more cheaper, this causes an inward shift in the demand curve, resulting in falling prices.

22. Consider the augmented matrix

$$\begin{bmatrix} 3 & 2 & 5 & | & 8 \\ 2 & 1 & 2 & | & 7 \\ 8 & 5 & 12 & | & 23 \end{bmatrix}$$

with row echelon form:

$$\begin{bmatrix} 1 & 0 & -1 & | & 6 \\ 0 & 1 & 4 & | & -5 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

These equations are consistent. They are not independent because the coefficient matrix is not of full rank. This system of equations will have infinitely many solutions given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$$