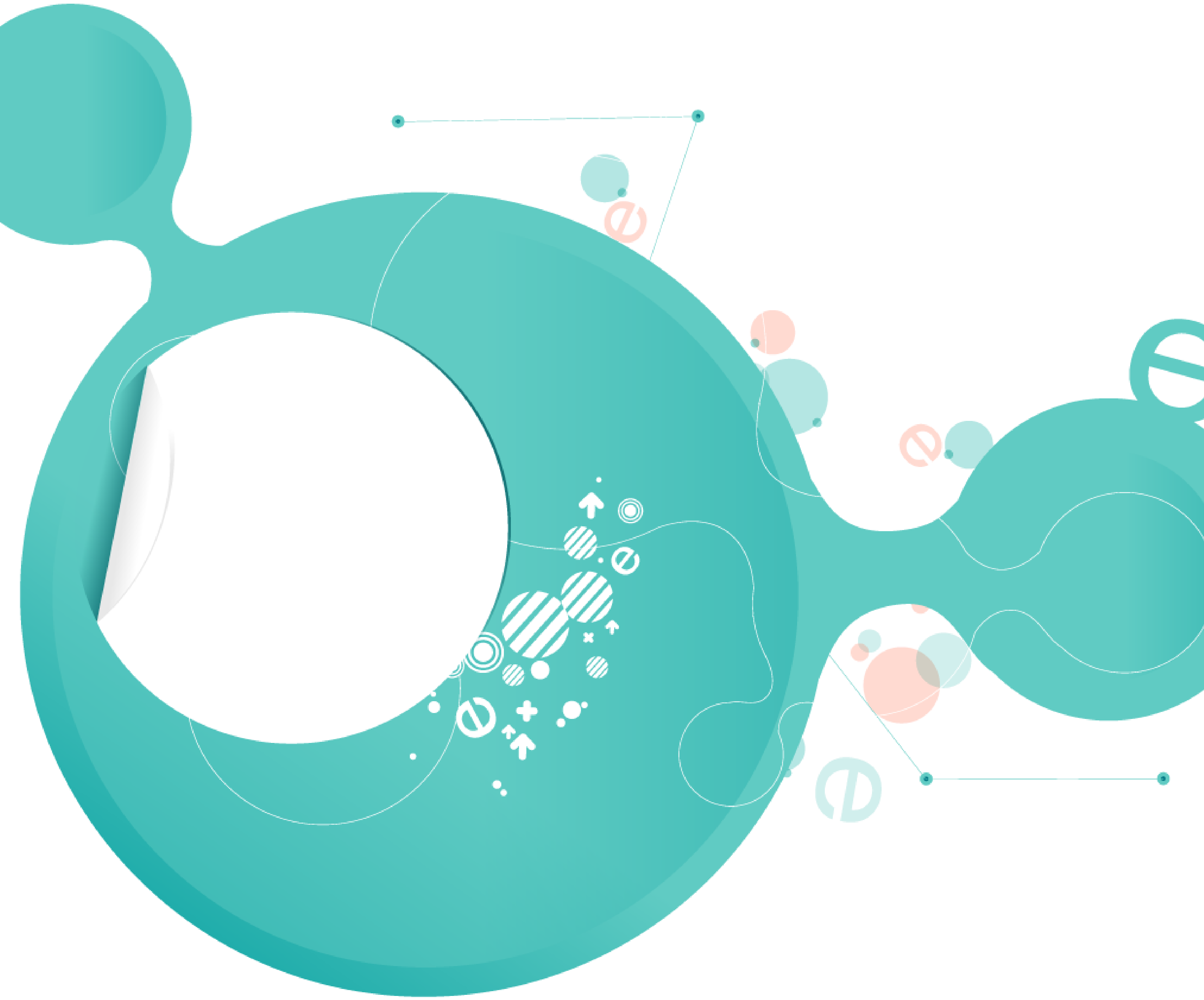
Mathematics for Economists

Tutorial Questions – Dynamic Analysis



Tutorial 5: Dynamic Analysis

ECO4112F 2011

1 First order differential and difference equations

Pemberton and Rau, Chapter 21 Exercises

1. Simple and separable differential equations: 21.1.1 - 21.1.5 (p 416-417)
2. Linear equations with constant coefficients: 21.2.1 - 21.2.6 (p 422-423)
3. Harder first-order equations: 21.3.1 - 21.3.4 (p 428)
4. Difference equations: 21.4.1 - 21.4.6 (p 435-436)
5. Try Problems 21-1 - 21-4 for something more interesting. (p 436-438)

Chiang, Chapter 15 Exercises
Exercise 15.1 (p 479)

3. Find the solution of each of the following:

   (a) $\frac{dy}{dt} + y = 4; y(0) = 0$

   (b) $\frac{dy}{dt} = 23; y(0) = 1$

   (c) $\frac{dy}{dt} - 5y = 0; y(0) = 6$

   (d) $\frac{dy}{dt} + 3y = 2; y(0) = 4$

   (e) $\frac{dy}{dt} - 7y = 7; y(0) = 7$

   (f) $3\frac{dy}{dt} + 6y = 5; y(0) = 0$

Exercise 15.3 (p 486)

Solve the following first-order linear differential equations; if an initial condition is given, definitise the arbitrary constant:

1. $\frac{dy}{dt} + 5y = 15$

2. $\frac{dy}{dt} + 2ty = 0$

3. $\frac{dy}{dt} + 2ty = t; y(0) = \frac{3}{2}$

4. $\frac{dy}{dt} + t^2y = 5t^2; y(0) = 6$

5. $2\frac{dy}{dt} + 12y + 2e^t = 0; y(0) = \frac{6}{7}$

6. $\frac{dy}{dt} + y = t$

Exercise 15.5 (p 495)

1. Solve the following differential equations:

   (a) $2tdy + 2ydt = 0$

   (b) $\frac{y}{y+t}dy + \frac{2t}{y+t}dt = 0$

   (c) $\frac{dy}{dt} = -\frac{t}{y}$

   (d) $\frac{dy}{dt} = 3y^2t$

2 Trigonometric functions and complex numbers

Chiang, Chapter 16 Exercises

Complex numbers and circular functions: Exercise 16.2, 1-8 (p 521-522)

Pemberton and Rau, Chapter 22 Exercises

1. Cycles, circles and trigonometry: 22.1.1, 22.1.2, and 22.1.4 (p 445)

2. Extending the definitions: 22.2.1 - 22.2.4 (p 451-452)

3. Calculus with circular functions: 23.3.1, 23.3.3 and 23.3.5 (p 457)

4. Polar coordinates: 22.4.1 - 22.4.3 (p 460)

Pemberton and Rau, Chapter 23 Exercises

1. The complex number system: 23.1.1 - 23.1.6 (p 466)
2. The trigonometic form: 23.2.1 - 23.2.5 (p 470 - 471)
3. Complex exponentials and polynomials: 23.3.1 - 23.3.3 (p 476)

3 Second order differential and difference equations

Pemberton and Rau, Chapter 24 Exercises

1. Second order differential equations: 24.1.1 - 21.1.4 (p 486)
2. Qualitative behaviour: 24.2.1 - 24.2.3 (p 493)
3. Second-order difference equations: 24.3.1 - 24.3.5 (p 501)
4. Try Problems 24-1 - 24-4 for something more interesting. (p 502-503)

Chiang, Chapter 16 Exercises

Exercise 16.1 (p 511)

1. Find the particular integral of each equation:
   (a) $y''(t) - 2y'(t) + 5y = 2$
   (b) $y''(t) + y'(t) = 7$
   (c) $y''(t) + 3y = 9$
   (d) $y''(t) + 2y'(t) - y = -4$
   (e) $y''(t) = 12$
2. Find the complementary function of each equation:
   (a) $y''(t) + 3y'(t) - 4y = 12$
   (b) $y''(t) + 6y'(t) + 5y = 10$
   (c) $y''(t) - 2y'(t) + y = 3$
   (d) $y''(t) + 8y'(t) + 16y = 0$
3. Find the general solution of each differential equation in question 2, and then definitise the solution with the initial conditions $y(0) = 4$ and $y'(0) = 2$.

4 Mixed bag

Solve the following differential equations. Where boundary conditions are given, make sure you solve for the specific solution by establishing the value of all arbitrary constants.

1. $\dfrac{d^3y}{dx^3} = 8$

2. $y'' = x^2 + 5x - 12$

3. $y'' = 6x + 6$
   $y(0) = 8$
   $y'(0) = 5$

4. $\dfrac{d^2y}{dt^2} = e^t + 2e^{-t}$
   $y(0) = 3$
   $y'(0) = -1$

5. $y'' = -x^{-2} + 6x$
   $y(1) = 5$
   $y'(1) = 4$

6. $\dfrac{dy}{dx} - 1 = e^{2x}$

7. $(1 + x)\,dy - ydx = 0$

8. $\dfrac{dy}{dx} = -\dfrac{x}{y}$
   $y(4) = 3$

9. $xy^4dx + \left(y^2 + 2\right) e^{-3x}dy = 0$

10. $y^{-1/2}dy = xdx$

11. $\dfrac{dy}{dx} = \sin(5x)$

12. $dx + e^{3x}dy = 0$

13. $xy' = 4y$

14. $\dfrac{dx}{dy} = \dfrac{x^2y^2}{1 + x}$

15. $\dfrac{dy}{dx} = e^{3x+2y}$

16. $x\dfrac{dy}{dx} - 4y = x^6 e^x$

17. $\dfrac{dy}{dx} - 3y = 0$

18. $\left(x^2 + 9\right)\dfrac{dy}{dx} + xy = 0$

19. $x\dfrac{dy}{dx} + y = 2x$
    $y(1) = 0$

20. $\dfrac{dy}{dx} + 5y = 20$
    $y(0) = 2$

21. $(x+1)\dfrac{dy}{dx} + y = \ln x$
    $y(1) = 10$

22. $x\dfrac{dy}{dx} + (3x+1)\,y = e^{-3x}$

5 Eigenvalues and eigenvectors

Question 1

For each of the matrices below:

(a) Find the eigenvalues and eigenvectors.

(b) Find a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{S}$ such that $\mathbf{A} = \mathbf{SDS}^{-1}$.

(c) Find a general expression for $\mathbf{A}^k$ for every positive integer k.

1. $\mathbf{A} = \begin{bmatrix} 4 & 4 \\ -3 & -3 \end{bmatrix}$

2. $\mathbf{A} = \begin{bmatrix} 5 & 3 \\ -4 & -2 \end{bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} -2 & -4 \\ 0 & -4 \end{bmatrix}$

4. $\mathbf{A} = \begin{bmatrix} 6 & -6 \\ 0 & 5 \end{bmatrix}$

5. $\mathbf{A} = \begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix}$

Question 2

For each of the matrices below:

(a) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

(b) Find the trace and determinant of $\mathbf{A}$.

(c) Comment on the definiteness of $\mathbf{A}$.

1. $\mathbf{A} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 5 \end{bmatrix}$

2. $\mathbf{A} = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

Question 3

1. Given the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 0 & 4 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 4 & 0 & 7 \end{bmatrix}$$

and three of the eigenvalues: $\lambda_1 = 5, \lambda_2 = 4$ and $\lambda_3 = 1$.

(a) Find the fourth eigenvalue, λ_4.

(b) Find the determinant of $\mathbf{A}$.

2. Given the matrix

$$\mathbf{A} = \begin{bmatrix} 67 & 266 & -30 & 64 \\ -24 & -91 & 12 & -20 \\ -6 & -42 & 10 & -12 \\ 42 & 126 & -21 & 21 \end{bmatrix}$$

and three of the eigenvalues: $\lambda_1 = -14, \lambda_2 = -7$ and $\lambda_3 = 7$.

(a) Find the fourth eigenvalue, λ_4.

(b) Find the determinant of $\mathbf{A}$.

Question 4

1. Find a 2×2 matrix $\mathbf{A}$ whose eigenvalues and corresponding eigenvectors are given by:

$$\lambda_1 = 1 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$
$$\lambda_2 = 2 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

2. Find a 3×3 matrix $\mathbf{A}$ whose eigenvalues and corresponding eigenvectors are given by:

$$\lambda_1 = 3 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$
$$\lambda_2 = -2 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$
$$\lambda_3 = 1 \quad \Rightarrow \quad \mathbf{x}_3 = \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}.$$

6 Dynamic Systems

Pemberton and Rau, Chapter 26 Exercises

1. 26.1.1 and 26.1.2 (p 534-535)
2. 26.2.1 and 26.2.2 (p 539)
3. 26.3.3 (p 550-551)

Tutorial 5: Dynamic Analysis

SELECTED SOLUTIONS

ECO4112F 2011

1 First order differential and difference equations

Pemberton and Rau, Chapter 21 Exercises

1. 21.1.1 $y = \frac{1}{4}t^4 + C$. The solution curves are U-shaped with vertex at $(0, C)$.

   (a) $y = \frac{1}{4}t^4 + 4$, (b) $y = \frac{1}{4}t^4 - 64$

   21.1.3 $y = 2\exp\left(\frac{3}{2}x^2\right)$

   21.1.5 Separating the variables and integrating,

   $$at = \int \frac{1}{y}dy + \int \frac{b}{a - by}dy = \ln y - \ln(a - by) + \text{constant.}$$

   Taking exponentials,

   $$e^{at} = \frac{Cy}{a - by} \tag{1}$$

   where C is a constant. Setting $t = 0$ in (1) gives $C = \frac{a - by_0}{y_0}$, and solving (1) for y yields the required solution. Since $C > 0$ by our assumptions on $y_0, 0 < y < \frac{a}{b}$. Since $a > 0, y \to \frac{a}{b}$ as $t \to \infty$.

2. 21.2.1 (a) $y = 2 + Ae^{-7t}; y \to 2$ as $t \to \infty$

   (b) $y = -2 + Ae^{7t}$. When $A > 0, y \to \infty$ as $t \to \infty$;
   when $A < 0, y \to -\infty$ as $t \to \infty$;when $A = 0, y = -2$ for all t.

   21.2.3 $y = 14 + Ae^{-t/7}, y = 14 - 9e^{-t/7}$

   21.2.5 $y = \frac{1}{9}\left(10e^{3x} - 12x - 1\right)$

3. 21.3.1 $y = 3e^{-t} + Ae^{-2t}$

   21.3.3 $y = \left[\frac{t}{4} + \frac{1}{32} + Ae^{8t}\right]^{-1/2}$

4. 21.4.1 Putting $\triangle y_t = 0$ gives the constant particular solution $Y_t = \frac{b}{a}$. In the text the equation is written in the form $y_{t+1} + (a-1)y_t = b$; the constant particular solution is obtainded by setting $y_{t+1} = y_t = Y$ and solving for Y. Putting $\triangle y_t = 0$ is equivalent to this but more directly analogous to finding the constant particular solution of a first order differential equation by setting $\frac{dy}{dt} = 0$.

21.4.3 (a) Not equivalent, $y_t = 3 + A\left(-\frac{1}{3}\right)^t$

(b) Equivalent, $y_t = 3 + A(-3)^t$

21.4.5 (a) $y_t = 2 + A\left(-\frac{5}{3}\right)^t$, $y_t = 2 - 2\left(-\frac{5}{3}\right)^t$

(b) $y_t = 2 + A\left(-\frac{3}{5}\right)^t$, $y_t = 2 - 2\left(-\frac{3}{5}\right)^t$

Chiang, Chapter 15 Exercises

Exercise 15.1 (p 479)

3. (a) $y(t) = 4\left(1 - e^{-t}\right)$, (c) $y(t) = 6e^{5t}$, (e) $y(t) = 8e^{7t} - 1$

Exercise 15.3 (p 486)

1. $y(t) = Ae^{-5t} + 3$
2. ...
3. $y(t) = Ae^{-t^2} + \frac{1}{2}$
4. ...
5. $y(t) = e^{-6t} - \frac{1}{7}e^t$
6. *Hint*: $\int xe^x dx = e^x(x-1) + C$

Exercise 15.5 (p 495)

1. (c) $y(t) = \left(A - t^2\right)^{1/2}$

2 Trigonometric functions and complex numbers

Pemberton and Rau, Chapter 22 Exercises

1. 22.1.1 (a) 0.175, (b) 1.484, (c) 0.332

2. 22.2.1 Sines: $\frac{1}{\sqrt{2}}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{2}, -\frac{\sqrt{3}}{2}.$

   Cosines: $-\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}$

   Tangents: $-1, \frac{1}{\sqrt{3}}, -\sqrt{3}, \frac{1}{\sqrt{3}}, -\sqrt{3}$

   22.2.3 $\frac{1}{\sqrt{2}}, \frac{\pi}{4}.$

3. 23.3.1 (a) $a\cos ax$, (b)$-a\sin ax$, (c) $\frac{a}{\cos^2 ax}$, (d) $5\sin^4 x\cos x$, (e) $5x^4\cos\left(x^5\right)$,

   (f) $\sin x + x\cos x$, (g) $5x^4\tan 2x + \frac{2x^5}{\cos^2 2x}$, (h) $-\frac{(x\sin x+\cos x)}{x^2}$

   23.3.3 (a) $\frac{\sin^7 x}{7} + A$, (b) $\frac{\pi+4}{4\sqrt{2}} - 1$

   23.3.5 (a) $\frac{\pi}{3}$, (b) $\frac{2\pi}{3}$, (c) $\frac{\pi}{4}$, (d) $-\frac{\pi}{6}$.

4. 22.4.1 (a) $\left(2, \frac{\pi}{3}\right)$, (b) $\left(\sqrt{8}, \frac{3\pi}{4}\right)$, (c) $\left(1, -\frac{2\pi}{3}\right)$, (d) $\left(\sqrt{2}, -\frac{\pi}{4}\right)$

   22.4.3 (a) Circle of radius 2 and centre $(0,0)$

   (b) Straight line parallel to y-axis, 4 units to the right of it.

   (c) Straight line parallel to x-axis, 3 units above it.

Pemberton and Rau, Chapter 23 Exercises

1. 23.1.1 $1, -32i, -1, i, -i$

   23.1.3 (a) $-2 \pm 3i$, (b) $\frac{1}{2}\left(5 \pm i\sqrt{11}\right)$

   23.1.5 Let $v = w/z$. Then $vz = w$, so $|v|\,|z| = |w|$, whence $|v| = |w|\,/\,|z|$.

2. 23.2.3 (a) $1 + i\sqrt{3}, 2, \frac{\pi}{3}, 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

   (b) $-2 + 2i, 2\sqrt{2}, \frac{3\pi}{4}, 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(c) $-\frac{1}{2}\left(1+i\sqrt{3}\right), 1, -\frac{2\pi}{3}, \cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)$

(d) $1-i, \sqrt{2}, -\frac{\pi}{4}, \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$

23.2.5 -2^{20}

3. 23.3.1 $\sqrt{2}e^{\pi i/4}, \sqrt{2}e^{3\pi i/4}, 2e^{\pi i/3}, e^{-2\pi i/3}$

23.3.3 $(1+2i)\,z^2+(3-i)\,z-4-3i$

23.3.5 $3i \pm (1+i)\sqrt{2}$

3 Second order differential and difference equations

Pemberton and Rau, Chapter 24 Exercises

1. 24.1.1 Setting $y = te^{pt}$ we have $\frac{dy}{dt} = (1+pt)\,e^{pt}$, whence $\frac{d^2y}{dt^2} = \left(2p+p^2t\right)e^{pt}$. Therefore

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = (2p+b)\,e^{pt} + \left[p^2+bp+c\right]te^{pt}$$

By definition of $p, b = -2p$ and the term in square brackets is zero; hence the differential equation is satisfied.

24.1.3 (a) $y = Ae^{t/3} + Be^{-t} - 6$

(b) $y = (At+B)\,e^{-3t} - \frac{3t-5}{27}$

2. 24.2.1 (a) **UN**, (b) **UN,** (c) **UO,** (d) **UO,** (e) **SN,** (f) **SN.**

24.2.3 (a) $\theta < 2\sqrt{\frac{\alpha}{\beta}}$,

(b) $\sigma < 2\sqrt{\frac{\beta}{\alpha}}$

3. 24.3.1 (a) $y_t = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^t - \left(\frac{1-\sqrt{5}}{2}\right)^t\right]$

(b) $y_t = \frac{2}{3}\left((-4)^t + 2^{1+t}\right)$

(c) $y_t = \frac{1}{3}\left(5\times 2^t - 2^{1-t}\right)$

24.3.3 $y_t = C\cos\left(\frac{2}{3}\pi t + \alpha\right) + \frac{4}{3}t - 1$

24.3.5 $Y_t - 2Y_{t-1} + \frac{4}{3}Y_{t-2} = 20, 60$

$$Y_t = C\left(\frac{4}{3}\right)^t \cos\left(\frac{1}{6}\pi t + \alpha\right) + 60, Y_t \to \infty \text{ as } t \to \infty.$$

Chiang, Chapter 16 Exercises

Exercise 16.1 (p 511)

1. (a) $y_P = \frac{2}{5}$, (c) $y_P = 3$, (e) $y_P = 6t^2$
2. (a) $y(t) = 6e^t + e^{-4t} - 3$ (c) $y(t) = e^t + te^t + 3$

4 Mixed bag

1. ...
2. ...
3. $y = x^3 + 3x^2 + 5x + 8$
4. $y = e^t + 2e^{-t}$
5. $y = \ln x + x^3 + 4$
6. $y = x + \frac{1}{2}e^{2x} + C$
7. $y = (1+x)\,e^C$ or $y = (1+x)\,A$
8. $\frac{y^2}{2} = -\frac{x^2}{2} + 12.5$
9. $\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} = \frac{1}{y} + \frac{2}{3y^3} + C$
10. $2y^{1/2} = \frac{x^2}{2} + C$
11. $y = -\frac{\cos(5x)}{5} + C$
12. $y = \frac{1}{3}e^{-3x} + C$

13. $y = cx^4$

14. $-3 + 3x \ln x = xy^3 + Cx$

15. $-3e^{-2y} = 2e^{3x} + C$

16. $y = x^5 e^x - x^4 e^x + Cx^4$

17. $y = Ce^{3x}$

18. $y = \dfrac{C}{\sqrt{x^2+9}}$

19. $y = x - \dfrac{1}{x}, x > 0$

20. $y = 4 - 2e^{-5x}$

21. $(x+1)\,y = x \ln x - x + 21, x > 0$

22. $y = e^{-3x} + \dfrac{C}{x} e^{-3x}, x > 0$

5 Eigenvalues and eigenvectors

Question 1

1. $\mathbf{A} = \begin{bmatrix} 4 & 4 \\ -3 & -3 \end{bmatrix}$

   (a) Eigenvalues: $1, 0$

   Eigenvectors: The non-zero multiples of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are the eigenvectors associated with the eigenvalue 0, and the non-zero multiples of $\begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}$ are the eigenvectors associated with the eigenvalue 1.

   (b)

$$\begin{aligned} \mathbf{D} &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathbf{S} &= \begin{bmatrix} -1 & -\frac{4}{3} \\ 1 & 1 \end{bmatrix} \end{aligned}$$

(c)

$$\mathbf{S}^{-1} = \begin{bmatrix} 3 & 4 \\ -3 & -3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{A}^k &= \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1} \\ &= \begin{bmatrix} -1 & -\frac{4}{3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -3 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 4 \\ -3 & -3 \end{bmatrix} \end{aligned}$$

2. $\mathbf{A} = \begin{bmatrix} 5 & 3 \\ -4 & -2 \end{bmatrix}$

   (a) Eigenvalues: $2, 1$

   Eigenvectors: The non-zero multiples of $\begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix}$ are the eigenvectors associated with the eigenvalue 1, and the non-zero multiples of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are the eigenvectors associated with the eigenvalue 2.

   (b)

$$\begin{aligned} \mathbf{D} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ \mathbf{S} &= \begin{bmatrix} -\frac{3}{4} & -1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

   (c)

$$\mathbf{S}^{-1} = \begin{bmatrix} 4 & 4 \\ -4 & -3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{A}^k &= \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1} \\ &= \begin{bmatrix} -\frac{3}{4} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -4 & -3 \end{bmatrix} \end{aligned}$$

3. $\mathbf{A} = \begin{bmatrix} -2 & -4 \\ 0 & -4 \end{bmatrix}$

   (a) Eigenvalues: $-2, -4$

Eigenvectors: The non-zero multiples of $\begin{bmatrix}2\\1\end{bmatrix}$ are the eigenvectors associated with the eigenvalue -4, and the non-zero multiples of $\begin{bmatrix}1\\0\end{bmatrix}$ are the eigenvectors associated with the eigenvalue -2.

(b)

$$\begin{aligned}\mathbf{D} &= \begin{bmatrix}-4 & 0\\ 0 & -2\end{bmatrix}\\ \mathbf{S} &= \begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}\end{aligned}$$

(c)

$$\mathbf{S}^{-1} = \begin{bmatrix}0 & 1\\ 1 & -2\end{bmatrix}$$

$$\begin{aligned}\mathbf{A}^k &= \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1}\\ &= \begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}(-4)^k & 0\\ 0 & (-2)^k\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & -2\end{bmatrix}\end{aligned}$$

4. $\mathbf{A} = \begin{bmatrix} 6 & -6\\ 0 & 5\end{bmatrix}$

(a) Eigenvalues: $6, 5$

Eigenvectors: The non-zero multiples of $\begin{bmatrix}6\\1\end{bmatrix}$ are the eigenvectors associated with the eigenvalue 5, and the non-zero multiples of $\begin{bmatrix}1\\0\end{bmatrix}$ are the eigenvectors associated with the eigenvalue 6.

(b)

$$\begin{aligned}\mathbf{D} &= \begin{bmatrix}5 & 0\\ 0 & 6\end{bmatrix}\\ \mathbf{S} &= \begin{bmatrix}6 & 1\\ 1 & 0\end{bmatrix}\end{aligned}$$

(c)

$$\mathbf{S}^{-1} = \begin{bmatrix}0 & 1\\ 1 & -6\end{bmatrix}$$

$$\begin{aligned}
\mathbf{A}^k &= \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1} \\
&= \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 6^k \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -6 \end{bmatrix}
\end{aligned}$$

5. $\mathbf{A} = \begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix}$

   (a) Eigenvalues: $15, -5$

   Eigenvectors: The non-zero multiples of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are the eigenvectors associated with the eigenvalue -5, and the non-zero multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are the eigenvectors associated with the eigenvalue 15.

   (b)

$$\begin{aligned}
\mathbf{D} &= \begin{bmatrix} -5 & 0 \\ 0 & 15 \end{bmatrix} \\
\mathbf{S} &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

   (c)

$$\mathbf{S}^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{aligned}
\mathbf{A}^k &= \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1} \\
&= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-5)^k & 0 \\ 0 & 15^k \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

Question 2

1. (a)

$$\begin{aligned}
\lambda\mathbf{I} - \mathbf{A} &= \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 5 \end{bmatrix} \\
&= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 5 \end{bmatrix} \\
&= \begin{bmatrix} \lambda - 2 & -1 & 2 \\ -1 & \lambda - 2 & 2 \\ 2 & 2 & \lambda - 5 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
\det(\lambda \mathbf{I} - \mathbf{A}) &= \begin{vmatrix} \lambda - 2 & -1 & 2 \\ -1 & \lambda - 2 & 2 \\ 2 & 2 & \lambda - 5 \end{vmatrix} \\
&= (\lambda - 2)(-1)^{1+1} \begin{vmatrix} \lambda - 2 & 2 \\ 2 & \lambda - 5 \end{vmatrix} + (-1)(-1)^{1+2} \begin{vmatrix} -1 & 2 \\ 2 & \lambda - 5 \end{vmatrix} \\
&\quad + (2)(-1)^{1+3} \begin{vmatrix} -1 & \lambda - 2 \\ 2 & 2 \end{vmatrix} \\
&= (\lambda - 2)[(\lambda - 2)(\lambda - 5) - (2)(2)] + (-1)(\lambda - 5) - (2)(2) \\
&\quad +2[(-1)(2) - (\lambda - 2)(2)] \\
&= (\lambda - 2)\left(\lambda^2 - 7\lambda + 10 - 4\right) - \lambda + 5 - 4 + 2(-2 - 2\lambda + 4) \\
&= (\lambda - 2)\left(\lambda^2 - 7\lambda + 6\right) - \lambda + 1 + 2(-2\lambda + 2) \\
&= \lambda^3 - 2\lambda^2 - 7\lambda^2 + 14\lambda + 6\lambda - 12 - \lambda + 1 - 4\lambda + 4 \\
&= \lambda^3 - 9\lambda^2 + 15\lambda - 7 \\
&= (\lambda - 1)(\lambda^2 - 8\lambda + 7) \\
&= (\lambda - 1)(\lambda - 1)(\lambda - 7) \\
&= (\lambda - 1)^2 (\lambda - 7)
\end{aligned}
$$

We find the eigenvalues by setting the characteristic polynomial equal to zero and solving for λ:

$$
\begin{aligned}
\det(\lambda \mathbf{I} - \mathbf{E}) &= 0 \\
(\lambda - 1)^2 (\lambda - 7) &= 0 \\
\lambda &= 1; 1; 7
\end{aligned}
$$

The eigenvectors associated with the eigenvalue 1 are the non-zero vectors $\mathbf{x}$ such that

$$
\begin{aligned}
\mathbf{A}\mathbf{x} &= 1\mathbf{x} \\
\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 5 \end{bmatrix} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}
$$

Put this into equation form:

$$
\begin{aligned}
2x_1 + x_2 - 2x_3 &= x_1 \\
\Rightarrow x_1 + x_2 - 2x_3 &= 0
\end{aligned}
$$

and

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= x_2 \\ \Rightarrow x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

and

$$\begin{aligned} -2x_1 - 2x_2 + 5x_3 &= x_3 \\ \Rightarrow -2x_1 - 2x_2 + 4x_3 &= 0 \\ \Rightarrow x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

Because the eigenvalue 1 has multiplicity 2 (i.e. it is repeated twice) we must find 2 linearly independent eigenvectors associated with this eigenvalue. This means that the 2 eigenvectors must satisfy the condition $x_1 + x_2 - 2x_3 = 0$ and be linearly independent (i.e. they must not be multiples of each other).

Therefore, the eigenvectors corresponding to the eigenvalue 1 are the non-zero multiples of $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ .and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

The eigenvectors associated with the eigenvalue 7 are the non-zero vectors $\mathbf{x}$ such that

$$\begin{aligned} \mathbf{Ax} &= 7\mathbf{x} \\ \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 5 \end{bmatrix} &= 7 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

Put this into equation form:

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= 7x_1 \\ \Rightarrow -5x_1 + x_2 - 2x_3 &= 0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 7x_2 \\ \Rightarrow x_1 - 5x_2 - 2x_3 &= 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned} -2x_1 - 2x_2 + 5x_3 &= 7x_3 \\ \Rightarrow -2x_1 - 2x_2 - 2x_3 &= 0 \\ \Rightarrow x_1 + x_2 + x_3 &= 0 \end{aligned} \tag{4}$$

Because the eigenvalue 7 has multiplicity 1 (i.e. it is not repeated) we need to find only 1 eigenvector associated with this eigenvalue.
Now, from (4) :

$$x_3 = -(x_1 + x_2) \tag{5}$$

Substitute (5) into (3):

$$\begin{aligned} x_1 - 5x_2 - 2[-(x_1 + x_2)] &= 0 \\ \Rightarrow 3x_1 - 3x_2 &= 0 \\ \Rightarrow x_1 &= x_2 \end{aligned}$$

Substitute (5) into (2):

$$\begin{aligned} -5x_1 + x_2 - 2[-(x_1 + x_2)] &= 0 \\ \Rightarrow -3x_1 + 3x_2 &= 0 \\ \Rightarrow x_1 &= x_2 \end{aligned} \tag{6}$$

Use the result from (6) to simplify (5):

$$\begin{aligned} x_3 &= -(x_1 + x_1) \\ &= -2x_1 \\ &= -2x_2 \end{aligned} \tag{7}$$

Thus, the eigenvectors corresponding to the eigenvalue 7 must satisfy the conditions given in equations (6) and (7) above, i.e.

$$\begin{aligned} x_1 &= x_2 \\ x_3 &= -2x_1 = -2x_2 \end{aligned}$$

Therefore, the eigenvectors corresponding to the eigenvalue 7 are the non-zero multiples of $\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$.

(b)

$$\begin{aligned} \text{tr}\,\mathbf{A} &= \lambda_1 + \lambda_2 + \lambda_3 \\ &= 1 + 1 + 7 \\ &= 9 \end{aligned}$$

and

$$\begin{aligned} \det \mathbf{A} &= \lambda_1 \lambda_2 \lambda_3 \\ &= (1)(1)(7) \\ &= 7 \end{aligned}$$

(c) Positive definite.

2. (a)

$$\begin{aligned} \det(\lambda \mathbf{I} - \mathbf{A}) &= \begin{vmatrix} \lambda - 5 & -8 & -16 \\ -4 & \lambda - 1 & -8 \\ 4 & 4 & \lambda + 11 \end{vmatrix} \\ &\vdots \\ &= \lambda^3 + 5\lambda^2 + 3\lambda - 9 \\ &= (\lambda - 1)\left(\lambda^2 + 6\lambda + 9\right) \\ &= (\lambda - 1)(\lambda + 3)^2 \end{aligned}$$

Eigenvalues: $1; -3; -3$

The linearly independent eigenvectors corresponding to the eigenvalue -3 are the non-zero multiples of $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. (Remember that because the eigenvalue -3 has multiplicity 2 (i.e. it is repeated twice) we must find 2 linearly independent eigenvectors that correspond to this eigenvalue).

The eigenvectors corresponding to the eigenvalue 1 are the non-zero multiples of $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

(b)

$$\begin{aligned} \operatorname{tr} \mathbf{A} &= \lambda_1 + \lambda_2 + \lambda_3 \\ &= -3 - 3 + 1 \\ &= -5 \end{aligned}$$

and

$$\begin{aligned} \det \mathbf{A} &= \lambda_1 \lambda_2 \lambda_3 \\ &= (-3)(-3)(1) \\ &= 9 \end{aligned}$$

(c) Indefinite

3. (a)

$$\begin{aligned}
\det(\lambda\mathbf{I}-\mathbf{A}) &= \begin{vmatrix} \lambda-3 & 2 & -2 \\ 0 & \lambda-1 & 0 \\ 1 & -1 & \lambda \end{vmatrix} \\
&\vdots \\
&= \lambda^3-4\lambda^2+5\lambda-2 \\
&= (\lambda-1)\left(\lambda^2-3\lambda+2\right) \\
&= (\lambda-1)(\lambda-1)(\lambda-2) \\
&= (\lambda-1)^2(\lambda-2)
\end{aligned}$$

Eigenvalues: $1, 1, 2$

The linearly independent eigenvectors corresponding to the eigenvalue 1 are the non-zero multiples of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. (Remember that because the eigenvalue 1 has multiplicity 2 (i.e. it is repeated twice) we must find 2 linearly independent eigenvectors that correspond to this eigenvalue).

The eigenvectors corresponding to the eigenvalue 2 are the non-zero multiples of $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

(b)

$$\begin{aligned}
\text{tr } \mathbf{A} &= \lambda_1+\lambda_2+\lambda_3 \\
&= 1+1+2 \\
&= 4
\end{aligned}$$

and

$$\begin{aligned}
\det \mathbf{A} &= \lambda_1\lambda_2\lambda_3 \\
&= (1)(1)(2) \\
&= 2
\end{aligned}$$

(c) Positive definite

Question 3

1. (a) The trace is the sum of the diagonal entries:

$$\begin{aligned} \text{tr } \mathbf{A} &= \sum_{i=1}^{4} a_{ii} \\ &= 4+2+3+7 \\ &= 16 \end{aligned}$$

The trace is also equal to the sum of the eigenvalues:

$$\begin{aligned} \text{tr } \mathbf{A} &= \lambda_1+\lambda_2+\lambda_3+\lambda_4 \\ \Rightarrow 16 &= 5+4+1+\lambda_4 \\ \Rightarrow \lambda_4 &= 6 \end{aligned}$$

(b) The determinant is the product of the eigenvalues:

$$\begin{aligned} \det \mathbf{A} &= \lambda_1\lambda_2\lambda_3\lambda_4 \\ &= (5)\,(4)\,(1)\,(6) \\ &= 120 \end{aligned}$$

2. (a) $\lambda_4 = 21$

(b) $\det \mathbf{A} = 14\,406$

Question 4

1. Recall that

$$\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$$

where

$$\begin{aligned} \mathbf{D} &= \text{diag}\,(\lambda_1, ..., \lambda_n) \\ \mathbf{S} &= \begin{pmatrix} \mathbf{x}^1 & \dots & \mathbf{x}^n \end{pmatrix} \end{aligned}$$

Now

$$\begin{aligned} \mathbf{D} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ \mathbf{S} &= \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} \\ \mathbf{S}^{-1} &= \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} \end{aligned}$$

So

$$\begin{aligned}\mathbf{A} &= \mathbf{SDS}^{-1}\\ &= \begin{bmatrix}3 & 5\\ 4 & 7\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}7 & -5\\ -4 & 3\end{bmatrix}\\ &= \begin{bmatrix}-19 & 15\\ -28 & 22\end{bmatrix}\end{aligned}$$

2.

$$\begin{aligned}\mathbf{D} &= \begin{bmatrix}3 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{bmatrix}\\ \mathbf{S} &= \begin{bmatrix}-3 & -2 & -6\\ 2 & 1 & 3\\ 1 & 0 & 1\end{bmatrix}\\ \mathbf{S}^{-1} &= \begin{bmatrix}1 & 2 & 0\\ 1 & 3 & -3\\ -1 & -2 & 1\end{bmatrix}\end{aligned}$$

So

$$\begin{aligned}\mathbf{A} &= \mathbf{SDS}^{-1}\\ &= \begin{bmatrix}-3 & -2 & -6\\ 2 & 1 & 3\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}3 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 1 & 3 & -3\\ -1 & -2 & 1\end{bmatrix}\\ &= \begin{bmatrix}1 & 6 & -18\\ 1 & 0 & 9\\ 2 & 4 & 1\end{bmatrix}\end{aligned}$$

6 Dynamic Systems

Pemberton and Rau, Chapter 26 Exercises

1. 26.1.1 (a) $\mathbf{y}(t) = \left(\frac{1}{2}\right)^t c_1 \begin{bmatrix}5\\ -1\end{bmatrix} + \left(-\frac{1}{4}\right)^t c_2 \begin{bmatrix}2\\ -1\end{bmatrix}$

   $\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$

   (b) $\mathbf{x}(t) = \begin{bmatrix}18\\ -2\end{bmatrix} + \left(\frac{1}{2}\right)^t c_1 \begin{bmatrix}5\\ -1\end{bmatrix} + \left(-\frac{1}{4}\right)^t c_2 \begin{bmatrix}2\\ -1\end{bmatrix}$

   $\mathbf{x}(t) \to \begin{bmatrix}18\\ -2\end{bmatrix}$ as $t \to \infty$

2. 26.2.1 General solution is $\mathbf{y}(t) = c_1 e^{2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The boundary condition implies that $c_1 = 1, c_2 = 3$.

3. 26.3.3 (a) The eigenvalues are $1 + i$ and $1 - i$.

   (b) $a = p + q, b = i(p - q)$

   (c) From the differential equation of the system, $y = \dot{x} - x$. Substituting into the right hand side of this equation the solution for x given in (b), we obtain

   $$y = (-a \sin t + b \cos t) e^t$$

   Hence the general solution is

   $$\begin{bmatrix} x \\ y \end{bmatrix} = a e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + b e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$