Mathematics for Economists

Tutorial Questions – Comparative Statics

Tutorial 2: Comparative Statics

ECO4112F 2011

1 Derivatives and Rules of Differentiation

For each of the functions below:

(a) Find the difference quotient.

(b) Find the derivative $\frac{dy}{dx}$.

(c) Find $f'(4)$ and $f'(3)$.

1. $y = 4x^2 + 9$
2. $y = 5x^2 - 4x$
3. $y = 5x - 2$

Find the derivative of each of the following:

4. $y = f(x) = x^{13}$
5. $y = f(x) = 7x^6$
6. $y = f(u) = -4u^{-1/2}$
7. $y = f(x) = 63$
8. $y = f(x) = 3x^{-1}$
9. $y = f(x) = -x^{-4}$
10. $y = f(w) = 9w^4$
11. $y = f(u) = au^b$
12. $y = f(x) = cx^2$
13. $y = f(w) = \frac{3}{4}w^{4/3}$
14. $y = f(x) = ax^2+bx+c$
15. $y = 7x^4+2x^3-3x+37$
16. $y = \left(9x^2 - 2\right)(3x+1)$
17. $y = (3x+11)\left(6x^2 - 5x\right)$
18. $y = x^2(4x+6)$
19. $y = (ax-b)\left(cx^2\right)$
20. $y = (2-3x)(1+x)(x+2)$
21. $y = \left(x^2+3\right)x^{-1}$
22. $y = \frac{4x}{x+5}$
23. $y = \frac{x+7}{x}$
24. $y = \frac{ax^2+b}{cx+d}$
25. $y = \left(3x^2 - 13\right)^3$
26. $y = \left(8x^3 - 5\right)^9$
27. $y = (ax+b)^4$
28. $y = (16x+3)^{-2}$
29. $y = \frac{x^3+2x}{x+1}$
30. $y = \frac{5x^4}{x^2+4x}$
31. $y = \frac{x\left(1+x^2\right)^2}{\sqrt{2+x^2}}$
32. $y = \frac{x\left(\sqrt[3]{1+x^2}\right)}{\sqrt{2+x^2}}$
33. $y = f(t) = \ln at$
34. $y = f(t) = t^3 \ln t^2$
35. $y = e^{2t+4}$
36. $y = e^{t^2+1}$
37. $y = e^{ax^2+bx+c}$
38. $y = \ln 8t^5$
39. $y = \ln(t+9)$
40. $y = \ln x - \ln(1+x)$

Find the derivative of each of the following by first taking the natural log of both sides:

41. $y = \frac{x^2}{(x+3)(2x+1)}$

42. $y = \frac{3x}{(x+2)(x+4)}$

43. Given $y = u^3 + 1$ where $u = 5 - x^2$, find $\frac{dy}{dx}$.

44. Given $w = ay^2$ where $y = bx^2 + cx$, find $\frac{dw}{dx}$.

45. Are the following functions monotonic?

    (a) $y = -x^6 + 5 \ (x > 0)$

    (b) $y = 4x^5 + x^3 + 3x$

46. Given the function $f(x) = ax + b$, find the derivatives of

    (a) $f(x)$

    (b) $xf(x)$

    (c) $\frac{1}{f(x)}$

    (d) $\frac{f(x)}{x}$

47. Given the total cost function $C = Q^3 - 5Q^2 + 14Q + 75$, what is the variable cost function? Find the derivative of the variable cost function, and give the economic meaning of this derivative.

48. Given the average cost function $AC = Q^2 - 4Q + 214$, find the marginal cost function.

49. Given that average revenue is given by $AR = 60 - 3Q$, find the marginal revenue curve.

50. If $f(H, h) = \lambda(1 - \beta H)h$, where λ and β are positive constants, find f_H and f_h.

51. If $f(T, t) = TQn + TQnm(t - T) - T^2 rQn$, where Q, n, m and r are positive constants, find f_T and f_t.

Find $\dfrac{\partial y}{\partial x_1}$ and $\dfrac{\partial y}{\partial x_2}$ for each of the following functions:

52. $y = 2x_1^3 - 11x_1^2 x_2 + 3x_2^2$

53. $y = 7x_1 + 5x_1 x_2^2 - 9x_2^3$

54. $y = (2x_1 + 3)(x_2 - 2)$

55. $y = \dfrac{4x_1 + 3}{x_2 - 2}$

Find f_x and f_y for each of the following functions:

56. $f(x, y) = x^2 + 5xy - y^3$

57. $f(x, y) = \left(x^2 - 3y\right)(x - 2)$

58. $f(x, y) = \dfrac{2x - 3y}{x + y}$

59. $f(x, y) = \dfrac{x^2 - 1}{xy}$

60. If the utility function of an individual takes the form $U = U(x, y) = (x + 2)^2 (y + 3)^3$, where U is total utility and x and y are the quantities of the two commodities consumed.

    (a) Find the marginal utility of each of the two commodities.

    (b) Find the value of the marginal utility of commodity x when 3 units of each commodity are consumed.

61. Given the national income model below

$$\begin{aligned} Y &= C + I + G \\ C &= a + b(Y - T) & \quad a > 0, 0 < b < 1 \\ T &= d + tY & \quad d > 0, 0 < t < 1 \end{aligned}$$

    where Y is national income, C is (planned) consumption expenditure, I is investment expenditure, G is government expenditure and T is taxes.

    (a) Solve for Y^*, C^* and T^* using Cramer's Rule.

    (b) Find the government expenditure multiplier and the investment multiplier.

    (c) How is consumption affected by increased government spending. Do your findings accord with economic logic?

62. Let the national income model be

$$\begin{aligned} Y &= C + I + G \\ C &= a + b(Y - T) && a > 0, 0 < b < 1 \\ G &= gY && 0 < g < 1 \end{aligned}$$

where Y is national income, C is (planned) consumption expenditure, I is investment expenditure, G is government expenditure and T is taxes.

(a) Solve for Y^*, C^* and G^* and state what restriction is required on the parameters for a solution to exist.

(b) Give the economic meaning of the parameter g.

(c) Find the tax multiplier and the investment multiplier, and give the economic intuition behind their signs.

(d) Show that an increase in tax has a negative impact on consumption.

63. Consider the following simple Keynesian macroeconomic model

$$\begin{aligned} Y &= C + I + G \\ C &= 200 + 0.8Y \\ I &= 1000 - 2000R \end{aligned}$$

where Y is national income, C is (planned) consumption expenditure, I is investment expenditure, G is government expenditure and R is the interest rate.

Use Cramer's Rule to solve for Y^*, and evaluate the effect of a R50 billion decrease in government spending on national income.

64. Use Jacobian determinants to test the existence of functional dependence between the paired functions below:

(a) $y_1 = 3x_1^2 + x_2$
$y_2 = 9x_1^4 + 6x_1^2 (x_2 + 4) + x_2 (x_2 + 8) + 12$

(b) $y_1 = 3x_1^2 + 2x_2^2$
$y_2 = 5x_1 + 1$

2 General Function Models (Total Differentiation)

1. Find the total differential dy, given:

   (a) $y = -x\left(x^2+3\right)$

   (b) $y = (x-8)(7x+5)$

   (c) $y = \dfrac{x}{x^2+1}$

   (d) $y = 10x_1^3 - 2x_2^3$

   (e) $y = 3x_1^2 + 4x_2^3$

   (f) $y = 3x^2 + xz - 2z^3$

   (g) $y = 2x_1 + 9x_1x_2 + x_2^2$

2. Given the consumption function $C = a + bY$ (where $a > 0, 0 < b < 1, Y > 0$), find the income elasticity of consumption ε_{CY}.

3. Find the total derivative $\dfrac{dz}{dy}$, given

   (a) $z = f(x,y) = 2x + xy - y^2$ where $x = g(y) = 3y^2$

   (b) $z = f(x,y) = (x+y)(x-2y)$ where $x = g(y) = 2 - 7y$

4. Find the rate of change of output with respect to time, if the production function is $Q = A(t)K^\alpha L^\beta$, where $A(t)$ is an increasing function of t, $K = K_0e^{at}$ and $L = L_0 + bt$.

5. Given an isoquant which represents a production function $\overline{y} = f(x_1, x_2)$, where $\overline{y}$ is some constant level of output. Use total differentials to derive an expression for the marginal rate of technical substitution between the factors of production, x_1 and x_2.

3 Derivatives of Implicit Functions

1. Given $F(y,x) = x^3 - 2x^2y + 3xy^2 - 22 = 0$, is an implicit function $y = f(x)$ defined around the point $(y = 3, x = 1)$? If so, find $\frac{dy}{dx}$ by the implicit function rule, and evaluate it at this point.

2. Given $F(y,x) = 2x^2 + 4xy - y^4 + 67 = 0$, is an implicit function $y = f(x)$ defined around the point $(y = 3, x = 1)$? If so, find $\frac{dy}{dx}$ by the implicit function rule, and evaluate it at this point.

3. For each of the given equations $F(y,x) = 0$, find $\frac{dy}{dx}$ by (i) solving for y, (ii) the implicit function rule, and (iii) taking the total differential:

   (a) $F(y,x) = x^2 + xy - 14 = 0$

   (b) $F(y,x) = xy - 12 = 0$

   (c) $F(y,x) = 3x^3y - 2x^2 + 6 = 0$

4. Given $F(y,x) = x^2 - 3xy + y^3 - 7 = 0$, check that an implicit function $y = f(x)$ is defined around the point $(y = 3, x = 4)$. Hence find $\frac{dy}{dx}$ by the implicit function rule.

5. Given $F(y,x) = ye^y - x = 0$, verify the conditions of the implicit function theorem.

6. Consider the function corresponding to the upper semi-circle of the set of points in the xy-plane satisfying $x^2 + y^2 = 4$. Find $\frac{dy}{dx}$ by:

   (a) explicitly writing out the function $y = f(x)$ and finding its derivative.

   (b) the implicit function rule.

7. Given $F(y,x) = 10x_1y + x_1^2x_2 + y^2x_2 = 0$, find the value x_2 for which the implicit function $y = f(x_1, x_2)$ is defined around the point $(y = 1, x_1 = -2, x_2 =?)$? Hence find the values of $\frac{\partial y}{\partial x_1}$ and $\frac{\partial y}{\partial x_2}$ at this point.

8. Assuming that the equation $F(U, x, y) = 0$ implicitly defines a utility function $U = f(x, y)$, find an expression for the marginal rate of substitution.

9. The market for a single commodity is described by the following set of equations

$$\begin{aligned} Q_d &= Q_s \\ Q_d &= D(P,G) \\ Q_s &= S(P,N) \end{aligned}$$

where G is the price of substitutes and N is the price of inputs, and G and N are exogenously given. The following assumptions are imposed

$$\begin{aligned} \frac{\partial D}{\partial P} &< 0, \frac{\partial D}{\partial G} > 0 \\ \frac{\partial S}{\partial P} &> 0, \frac{\partial S}{\partial N} < 0 \end{aligned}$$

Use the implicit function theorem to show that the system implicitly defines the functions $P^*(G,N)$ and $Q^*(G,N)$. Hence use the implicit function rule to find and sign the derivatives $\frac{\partial P^*}{\partial G}, \frac{\partial Q^*}{\partial G}, \frac{\partial P^*}{\partial N}$ and $\frac{\partial Q^*}{\partial N}$.

10. Consider the system of equations

$$\begin{aligned} F^1(x,y;a) &\equiv x^2 + axy + y^2 - 1 = 0 \\ F^2(x,y;a) &\equiv x^2 + y^2 - a^2 + 3 = 0 \end{aligned}$$

around the point $(x=0, y=1, a=2)$. What is the impact of a change in a on x and y?

11. Let the national income model be written in the form

$$\begin{aligned} Y - C - I_0 - G_0 &= 0 \\ C - \alpha - \beta(Y - T) &= 0 \\ T - \gamma - \delta Y &= 0 \end{aligned}$$

where the endogenous variables are Y (national income), C ((planned) consumption expenditure) and T (taxation), and the exogenous variables are I_0 (investment expenditure) and G_0 (government expenditure).

(a) Find the government expenditure multiplier by the implicit function rule.

(b) Find the nonincome tax multiplier by the implicit function rule.

Tutorial 2: Comparative Statics

SELECTED SOLUTIONS

ECO4112F 2011

1 Derivatives and Rules of Differentiation

1. (a) $\frac{\Delta y}{\Delta x} = 8x_0 + 4\Delta x$ (b) $\frac{dy}{dx} = 8x$ (c) $f'(4) = 32, f'(3) = 24.$

2. (a) $\frac{\Delta y}{\Delta x} = 10x_0 + 5\Delta x - 4$ (b) $\frac{dy}{dx} = 10x - 4$ (c) $f'(4) = 36, f'(3) = 26.$

3. (a) $\frac{\Delta y}{\Delta x} = 5$ (b) $\frac{dy}{dx} = 5$ (c) $f'(4) = 5, f'(3) = 5.$

4. $f'(x) = 13x^{12}$
5. $f'(x) = 42x^5$
6. $f'(u) = -2u^{-1/2} = \frac{-2}{\sqrt{u}}$
7. $f'(x) = 0$
8. $f'(x) = -3x^{-2} = \frac{-3}{x^2}$
9. $f'(x) = 4x^{-5} = \frac{4}{x^5}$
10. $f'(w) = 36w^3$
11. $f'(u) = abu^{b-1}$
12. $f'(x) = 2cx$
13. $f'(w) = w^{1/3}$
14. $f'(x) = 2ax + b$
15. $\frac{dy}{dx} = 28x^3 + 6x^2 - 3$
16. $\frac{dy}{dx} = 3\left(27x^2 + 6x - 2\right)$
18. $\frac{dy}{dx} = 12x(x+1)$
20. $\frac{dy}{dx} = -x(9x+14)$
33. $f'(t) = \frac{1}{t}$
34. $f'(t) = 2t^2(1 + 3\ln t)$
35. $\frac{dy}{dt} = 2e^{2t+4}$
36. $\frac{dy}{dt} = 2te^{t^2+1}$
37. $\frac{dy}{dx} = (2ax+b)\, e^{ax^2+bx+c}$
38. $\frac{dy}{dt} = \frac{5}{t}$
39. $\frac{dy}{dt} = \frac{1}{t+9}$
40. $\frac{dy}{dx} = \frac{1}{x(1+x)}$

41. $\frac{dy}{dx} = \frac{x(7x+6)}{(x+3)^2(2x+1)^2}$

42. $\frac{dy}{dx} = \frac{3\left(8-x^2\right)}{(x+2)^2(x+4)^2}$

47. $VC = Q^3 - 5Q^2 + 14Q$ (Recall that to find variable costs, you drop the fixed costs)
$\frac{dVC}{dQ} = 3Q^2 - 10Q + 14$ This is the same as the marginal cost function. (Check this by differentiating the total cost function - you will get the same result)

64. (a) $|J| = 0$ The functions are dependent.
(b) $|J| = -20x_2$ The functions are independent.

2 General Function Models (Total Differentiation)

1.

(a) $dy = -3\left(x^2+1\right)dx$
(b) $dy = (14x - 51)\,dx$
(c) $dy = \left(\frac{1-x^2}{\left(x^2+1\right)^2}\right)dx$
(d) $dy = 30x_1^2dx_1 - 6x_2^2dx_2$
(e) $dy = 6x_1dx_1 + 12x_2^2dx_2$
(f) $dy = (6x+z)\,dx + \left(x - 6z^2\right)dz$
(g) $dy = (2+9x_2)\,dx_1 + (9x_1+2x_2)\,dx_2$

2.

$$\frac{dC}{dY} = b \text{ and } \frac{C}{Y} = \frac{a+bY}{Y}$$

$$\begin{aligned} \therefore \varepsilon_{CY} &= \frac{dC/dY}{C/Y} \\ &= \frac{b}{(a+bY)/Y} \\ &= \frac{bY}{(a+bY)} \end{aligned}$$

3.

(a) $\frac{dz}{dy} = 10y + 9y^2$
(b) $\frac{dz}{dy} = 102y - 30$

4.

$$\begin{aligned}
dQ &= \frac{\partial Q}{\partial A}dA + \frac{\partial Q}{\partial K}dK + \frac{\partial Q}{\partial L}dL \\
\frac{dQ}{dt} &= \frac{\partial Q}{\partial A}\frac{dA}{dt} + \frac{\partial Q}{\partial K}\frac{dK}{dt} + \frac{\partial Q}{\partial L}\frac{dL}{dt} \\
&= \left[K^{\alpha}L^{\beta}\right]\left[A'(t)\right] + \left[\alpha A(t)K^{\alpha-1}L^{\beta}\right]\left[aK_0e^{at}\right] + \left[\beta A(t)K^{\alpha}L^{\beta-1}\right][b] \\
&= \left[K^{\alpha}L^{\beta}\right]\left(A'(t) + a\alpha\frac{A}{K}K_0e^{at} + b\beta\frac{A}{L}\right)
\end{aligned}$$

5. We know that $\overline{y}$ will be constant along a given isoquant.

   The total differential is given by

$$d\overline{y} = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2$$

   Because $\overline{y}$ is constant, $d\overline{y} = 0$. Thus

$$\begin{aligned}
d\overline{y} = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 &= 0 \\
\Rightarrow \frac{\partial f}{\partial x_2}dx_2 &= -\frac{\partial f}{\partial x_1}dx_1 \\
\frac{dx_2}{dx_1} &= -\frac{\partial f/\partial x_1}{\partial f/\partial x_2} = MRTS
\end{aligned}$$

3 Derivatives of Implicit Functions

1. (a) Does the function F have continuous partial derivatives?

$$\begin{aligned}
F_y &= -2x^2 + 6xy \\
F_x &= 3x^2 - 4xy + 3y^2
\end{aligned}$$

   Yes.

   (b) Is $F_y \neq 0$ at a point satisfying $F(y,x) = 0$?

$$\begin{aligned}
F(y=3, x=1) &= (1)^3 - 2(1)^2(3) + 3(1)(3)^2 - 22 = 1 - 6 + 27 - 22 = 0 \\
\text{and } F_y(y=3, x=1) &= -2(1)^2 + 6(1)(3) = 16 \neq 0
\end{aligned}$$

   Therefore, an implicit function $y = f(x)$ is defined around the point $(y = 3, x = 1)$ and we can use the implict function rule to find

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\left(\frac{3x^2 - 4xy + 3y^2}{-2x^2 + 6xy}\right)$$

At the point $(y = 3, x = 1)$

$$\frac{dy}{dx} = -\left(\frac{3(1)^2 - 4(1)(3) + 3(3)^2}{-2(1)^2 + 6(1)(3)}\right) = -\frac{9}{8}$$

2. An implicit function $y = f(x)$ is defined around the point $(y = 3, x = 1)$ and we can use the implict function rule to find

$$\frac{dy}{dx} = -\left(\frac{4x + 4y}{4x - 4y^3}\right)\frac{2}{13}$$

at the point $(y = 3, x = 1)$.

4. (a) Does the function F have continuous partial derivatives?

$$\begin{aligned} F_y &= -3x + 3y^2 \\ F_x &= 2x - 3y \end{aligned}$$

Yes.

(b) Is $F_y \neq 0$ at a point satisfying $F(y, x) = 0$?

$$\begin{aligned} F(y = 3, x = 4) &= (4)^2 - 3(4)(3) + (3)^3 - 7 = 0 \\ \text{and } F_y(y = 3, x = 4) &= -3(4) + 3(3)^2 = 15 \neq 0 \end{aligned}$$

Therefore, an implicit function $y = f(x)$ is defined around the point $(y = 3, x = 4)$ and we can use the implict function rule to find

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\left(\frac{2x - 3y}{-3x + 3y^2}\right)$$

5. (a) Does the function F have continuous partial derivatives?

$$\begin{aligned} F_y &= (1 + y)e^y \\ F_x &= -1 \end{aligned}$$

Yes.

(b) Is $F_y \neq 0$ at a point satisfying $F(y, x) = 0$?
From (a) we can see that

$$F_y = 0 \Leftrightarrow y = -1$$

Thus we can find combinations of $(y_0, x_0), y_0 \neq -1$ that satisfy $F(y_0, x_0) = 0$ and $F_y(y_0, x_0) \neq 0$. For example, the point $(y = 1, x = e)$

$$\begin{aligned} F(y = 1, x = e) &= (1)\, e^1 - e = 0 \\ \text{and } F_y(y = 1, x = e) &= (1 + 1)\, e^1 = 2e \neq 0 \end{aligned}$$

Therefore, an implicit function $y = f(x)$ is defined around the point $(y = 1, x = e)$ and we can use the implict function rule to find

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\left(\frac{-1}{(1 + y)\, e^y}\right) = \frac{1}{(1 + y)\, e^y}$$

6. (a)

$$\frac{dy}{dx} = -x\left(4 - x^2\right)^{-1/2}$$

   (b)

$$\frac{dy}{dx} = -\frac{x}{y}$$

7. $x_2 = 4$

9. Rewrite the system as two equations by letting $Q = Q_d = Q_s$:

$$\begin{aligned} Q &= D(P, G) \\ Q &= S(P, N) \end{aligned}$$

or equivalently:

$$\begin{aligned} F^1(P, Q; G, N) &= D(P, G) - Q = 0 \\ F^2(P, Q; G, N) &= S(P, N) - Q = 0 \end{aligned}$$

Check the conditions for the implicit function theorem:

(a)

$$\begin{aligned} F_P^1 &= \frac{\partial D}{\partial P} \\ F_Q^1 &= -1 \\ F_G^1 &= \frac{\partial D}{\partial G} \\ F_N^1 &= 0 \end{aligned}$$

$$\begin{aligned} F_P^2 &= \frac{\partial S}{\partial P} \\ F_Q^2 &= -1 \\ F_G^2 &= 0 \\ F_N^2 &= \frac{\partial S}{\partial N} \end{aligned}$$

Therefore, continuous partial derivatives wrt all endogenous and exogenous variables exist.

(b)

$$\begin{aligned} |J| &= \begin{vmatrix} F_P^1 & F_Q^1 \\ F_P^2 & F_Q^2 \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial D}{\partial P} & -1 \\ \frac{\partial S}{\partial P} & -1 \end{vmatrix} \\ &= -\frac{\partial D}{\partial P} + \frac{\partial S}{\partial P} \\ &= \frac{\partial S}{\partial P} - \frac{\partial D}{\partial P} \\ &> 0 \end{aligned}$$

Therefore, $|J| \neq 0$

Conditions for implicit function satisfied, and so system implicitly defines the functions $P^*(G, N)$ and $Q^*(G, N)$.

Use the implicit function rule:

First, take the total differential of each equation:

$$
\begin{aligned}
dF^1 = F^1_P dp + F^1_Q dQ + F^1_G dG + F^1_N dN &= 0 \\
\Rightarrow \frac{\partial D}{\partial P} dp - 1dQ + \frac{\partial D}{\partial G} dG + 0 &= 0 \\
\frac{\partial D}{\partial P} dp - 1dQ &= -\frac{\partial D}{\partial G} dG \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
dF^2 = F^2_P dp + F^2_Q dQ + F^2_G dG + F^2_N dN &= 0 \\
\Rightarrow \frac{\partial S}{\partial P} dp - 1dQ + 0 + \frac{\partial S}{\partial N} dN &= 0 \\
\frac{\partial S}{\partial P} dP - 1dQ &= -\frac{\partial S}{\partial N} dN \qquad (2)
\end{aligned}
$$

Putting equations (1) and (2) in matrix form

$$
\begin{bmatrix} \frac{\partial D}{\partial P} & -1 \\ \frac{\partial S}{\partial P} & -1 \end{bmatrix} \begin{bmatrix} dP \\ dQ \end{bmatrix} = \begin{bmatrix} -\frac{\partial D}{\partial G} \\ 0 \end{bmatrix} dG + \begin{bmatrix} 0 \\ -\frac{\partial S}{\partial N} \end{bmatrix} dN \qquad (3)
$$

Note that the coefficient matrix is the Jacobian matrix J.

To find $\frac{\partial P^*}{\partial G}$ and $\frac{\partial Q^*}{\partial G}$ we partially differentiate with respect with G, holding N constant which implies that $dN = 0$. Setting $dN = 0$ and dividing through by dG in (3) gives:

$$
\begin{bmatrix} \frac{\partial D}{\partial P} & -1 \\ \frac{\partial S}{\partial P} & -1 \end{bmatrix} \begin{bmatrix} \frac{\partial P^*}{\partial G} \\ \frac{\partial Q^*}{\partial G} \end{bmatrix} = \begin{bmatrix} -\frac{\partial D}{\partial G} \\ 0 \end{bmatrix}
$$

(Note the *partial derivative* signs - we are differentiating with respect to G, holding N constant).

Use Cramer's rule to solve for $\frac{\partial P^*}{\partial G}$ and $\frac{\partial Q^*}{\partial G}$:

$$
\begin{aligned}
\frac{\partial P^*}{\partial G} &= \frac{\begin{vmatrix} -\frac{\partial D}{\partial G} & -1 \\ 0 & -1 \end{vmatrix}}{|J|} \\
&= \frac{\frac{\partial D}{\partial G}}{\frac{\partial S}{\partial P} - \frac{\partial D}{\partial P}} \\
&> 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Q^*}{\partial G} &= \frac{\begin{vmatrix} \frac{\partial D}{\partial P} & -\frac{\partial D}{\partial G} \\ \frac{\partial S}{\partial P} & 0 \end{vmatrix}}{|J|} \\
&= \frac{\frac{\partial D}{\partial G}\frac{\partial S}{\partial P}}{\frac{\partial S}{\partial P} - \frac{\partial D}{\partial P}} \\
&> 0
\end{aligned}
$$

To find $\dfrac{\partial P^*}{\partial N}$ and $\dfrac{\partial Q^*}{\partial N}$ we partially differentiate with respect with N, holding G constant which implies that $dG = 0$. Setting $dG = 0$ and dividing through by dN in (3) gives:

$$
\begin{bmatrix} \frac{\partial D}{\partial P} & -1 \\ \frac{\partial S}{\partial P} & -1 \end{bmatrix} \begin{bmatrix} \dfrac{\partial P^*}{\partial N} \\ \dfrac{\partial Q^*}{\partial N} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\partial S}{\partial N} \end{bmatrix}
$$

(Note the *partial derivative* signs - we are differentiating with respect to N, holding G constant).

Use Cramer's rule to solve for $\dfrac{\partial P^*}{\partial N}$ and $\dfrac{\partial Q^*}{\partial N}$:

$$
\begin{aligned}
\frac{\partial P^*}{\partial N} &= \frac{\begin{vmatrix} 0 & -1 \\ -\frac{\partial S}{\partial N} & -1 \end{vmatrix}}{|J|} \\
&= \frac{-\frac{\partial S}{\partial N}}{\frac{\partial S}{\partial P} - \frac{\partial D}{\partial P}} \\
&> 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Q^*}{\partial N} &= \frac{\begin{vmatrix} \frac{\partial D}{\partial P} & 0 \\ \frac{\partial S}{\partial P} & -\frac{\partial S}{\partial N} \end{vmatrix}}{|J|} \\
&= \frac{-\frac{\partial D}{\partial P}\frac{\partial S}{\partial N}}{\frac{\partial S}{\partial P} - \frac{\partial D}{\partial P}} \\
&< 0
\end{aligned}
$$

10. At the point $(x = 0, y = 1, a = 2)$, the equations $F^1 = 0$ and $F^2 = 0$ are satisfied.

The F^i functions possess continuous partial derivatives:

$$\begin{aligned} F^1_x &= 2x + ay \\ F^1_y &= ax + 2y \\ F^1_a &= xy \\ \\ F^2_x &= 2x \\ F^2_y &= 2y \\ F^2_a &= -2a \end{aligned}$$

Thus, if the Jacobian $|J| \neq 0$ at point $(x = 0, y = 1, a = 2)$ we can use the implicit function theorem.

First take the total differential of the system

$$\begin{aligned} (2x + ay)\, dx + (ax + 2y)\, dy + xyda &= 0 \\ 2xdx + 2ydy - 2ada &= 0 \end{aligned}$$

Moving the exogenous differential da to the RHS and writing in matrix form we get

$$\begin{bmatrix} 2x + ay & ax + 2y \\ 2x & 2y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} -xy \\ 2a \end{bmatrix} da$$

where the coefficient matrix of the LHS is the Jacobian

$$|J| = \begin{vmatrix} F^1_x & F^1_y \\ F^2_x & F^2_y \end{vmatrix} = \begin{vmatrix} 2x + ay & ax + 2y \\ 2x & 2y \end{vmatrix}$$

At the point $(x = 0, y = 1, a = 2)$,

$$|J| = \begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} = 4 \neq 0$$

Therefore the implicit function rule applies and

$$\begin{bmatrix} 2x + ay & ax + 2y \\ 2x & 2y \end{bmatrix} \begin{bmatrix} \left(\frac{\partial x}{\partial a}\right) \\ \left(\frac{\partial y}{\partial a}\right) \end{bmatrix} = \begin{bmatrix} -xy \\ 2a \end{bmatrix}$$

Use Cramer's rule to find an expression for $\frac{\partial x}{\partial a}$ at the point $(x = 0, y = 1, a = 2)$:

$$\begin{aligned}
\left(\frac{\partial x}{\partial a}\right) &= \frac{\begin{vmatrix} -xy & ax + 2y \\ 2a & 2y \end{vmatrix}}{|J|} \\
&= \frac{\begin{vmatrix} 0 & 2 \\ 4 & 2 \end{vmatrix}}{4} \\
&= -\frac{8}{4} \\
&= -2
\end{aligned}$$

Use Cramer's rule to find an expression for $\frac{\partial y}{\partial a}$ at the point $(x = 0, y = 1, a = 2)$:

$$\begin{aligned}
\left(\frac{\partial x}{\partial a}\right) &= \frac{\begin{vmatrix} 2x + ay & -xy \\ 2x & 2a \end{vmatrix}}{|J|} \\
&= \frac{\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}}{4} \\
&= \frac{8}{4} \\
&= 2
\end{aligned}$$

11. See Chiang, example 6, pages 203 - 204.

 (a)

$$\frac{\partial Y^*}{\partial G} = \frac{1}{1 - \beta + \beta\delta}$$

 (b)

$$\frac{\partial Y^*}{\partial \gamma} = \frac{-\beta}{1 - \beta + \beta\delta}$$