Mathematics for Economists

Integration, Differential and Difference Equations

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Honours Introductory Maths Course 2011
Integration, Differential and Difference Equations

Reading: Chiang Chapter 14

Note: These notes do not fully cover the material in Chiang, but are meant to supplement your reading in Chiang.

Thus far the optimisation you have covered has been static in nature, that is optimising the value of a function without any reference to time. In static optimisation and comparative static analysis we make the assumption that the process of economic adjustment leads to an equilibrium, and we then examine the effect of changes of the exogenous variables on the equilibrium values of the endogenous variables. With dynamic analysis, time is explicitly considered in the analysis. While we are not covering dynamic analysis at this point, certain mathematic tools are required for dynamic analysis, such as integration and differential equations. Without these tools, it becomes impossible to consider problems which are not static in nature. We will be covering both of these topics in a mainly mathematical way, leaving economic problems for a later date.

Integration is the reverse process of differentiation. If a function $F(x)$ has first derivative $f(x)$ then the integral of $f(x)$ will yield $F(x)$. The notation to denote integration is as follows:

$\int f(x)dx$, where the integral sign is an elongated S. $f(x)$ is referred to as the integrand, and the dx sign reminds us that we are integrating with respect to the variable *x*. We go through the following explanation to determine where this notation comes from.

Suppose we are given an arbitrary function $f(x)$ and asked to find the area of the curve between 2 points, for example the area under the curve *f(x)* between 5 and 10.

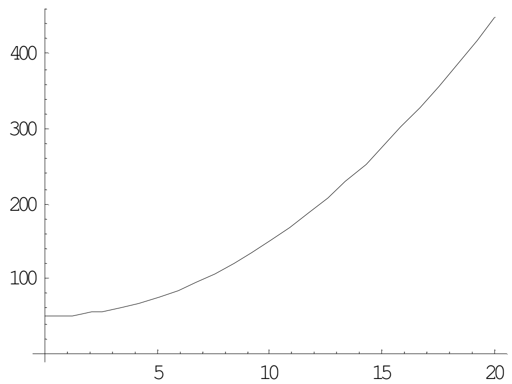

Figure 1.1

With a linear function, this equates to finding the area of a triangle and a rectangle as follows:

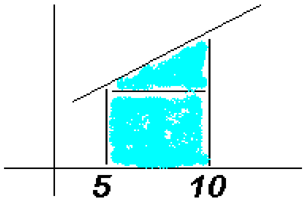

Figure 1.2

However with a non-linear function the problem becomes slightly more complex. What we can do, however, is attempt to find the area under the curve using a number of approximating rectangles as follows:

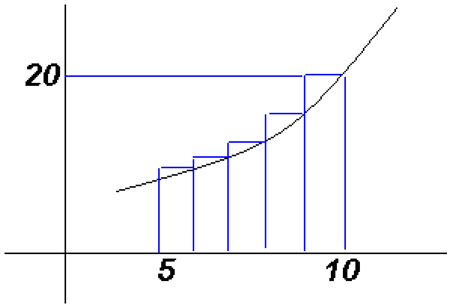

Figure 1.3

We let each of the rectangles have equal width and we call this width Δx. Each rectangle has a height equal to the function value, for example the height of the last rectangle where x=10 is equal to f(10)=20. Thus the area of the last rectangle is equal to 20(Δx), as area of a rectangle equals length times breadth, and here breadth is Δx and length is f(10)=20. The area of any of the rectangles is equal to length times breadth, which equals $f(x)$ times Δx, as all rectangles have equal breadth equal to Δx. To find the area under the curve we add up the area of each of the rectangles. This gives us the expression:

$$A = \sum_{i=1}^{n} f(x_i)\Delta x$$

Where
n = number of rectangles
x_i = the value of x at each point
$\sum$ = the sum of all the areas, starting from the first one (i = 1) and ending at the nth one (i = 1).

Obviously this sum will not be a very accurate representation of the area. But perhaps if we make our Δx smaller, then this expression will become a more accurate representation of the area under the curve, as there will be less overshooting by each rectangle. If initially we had ten rectangles, the area given by the sum of these rectangles' areas would obviously be more of an over-estimate (or maybe underestimate) than if we doubled the number of rectangles, and then summed their area. The more rectangles we use in this approximating process the better our estimate for the area under the curve.

For example, imagine we wish to find the area under the curve $f(x) = x^2$ between 0 and 1.

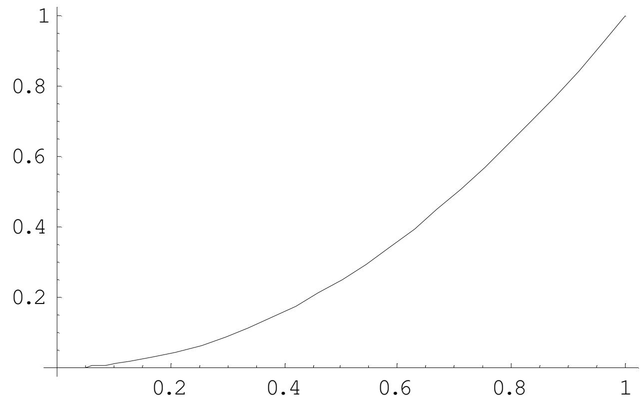

Figure 1.4

We could take four rectangles, each with a breadth Δx equal to 0.25, and take the heights from the right hand side of each rectangle. Hence the height for each rectangle will be:
$(0.25)^2$
$(0.5)^2$
$(0.75)^2$
1^2
Therefore the entire area equals to
$0.25(0.25)^2 + 0.25(0.5)^2 + 0.25(0.75)^2 + 0.25(1)^2 = 15/32 = 0.46875$

If we double the amount of rectangles from four to eight, we will use a Δx of 0.125, and the following right hand heights (remember the height of the rectangle is given by the function value *f*(*x*)).

$(0.125)^2$	$(0.25)^2$
$(0.375)^2$	$(0.5)^2$
$(0.625)^2$	$(0.75)^2$
$(0.875)^2$	1^2

The corresponding total area is given by the sum of each of the areas which is Δx multiplied by each function value: The final value we get is 0.3984375

As can be seen in figure 1.3, using right end points for the rectangles for an increasing function will give an over-estimate, while using right end points for a decreasing function will yield an over-estimate. Thus doubling the number of rectangles while trying to estimate the area under the graph $f(x)=x^2$ will begin to bring our estimate down to its true value. It appears that as the number of rectangles increases, our estimations become better and better approximations of the area. If we let the number of rectangles tend to infinity, we will obtain a perfectly accurate estimate for the area under our graph.

Our expression for the area under the curve now becomes:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

This gives us the expression for the definite integral, which gives us a way of finding the area under the continuous function *f*(*x*) between $x=a$ and $x=b$:

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

An explanation of the terminology:

$\int$ The integration sign is an elongated S, and was so chosen because an integral is a limit of sums.

a,b are the limits of integration,

a is the lower limit of integration

b is the upper limit of integration.

$f(x)$ is known as the integrand.

dx has no meaning by itself, but merely reminds us that we are integrating with respect to the variable *x*.

Fortunately when we want to find the area under a curve, we do not have to go into the long process of finding an expression for the sum of the area of *n* rectangles: a number of theorems make the process easier.

Before we set out the properties of the definite integral, some rules of integration are as follows: (see page 439 and onwards in Chiang for examples).

1. **The Power Rule**

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c \quad (n \neq -1)$$

2. **The Exponential Rule**

$$\int e^x dx = e^x + c$$

3. **The Logarithmic Rule**

$$\int \frac{1}{x} dx = \ln x + c \quad (x > 0)$$

Properties of the definite integral:

$$1. \int_a^b c\,dx = c(b-a)$$

$$2. \int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$3. \int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$4. \int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$5. \int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

Property 1 states that the integral of a constant function *y*=*c* is the constant times the length of the interval, as seen in figure 1.5

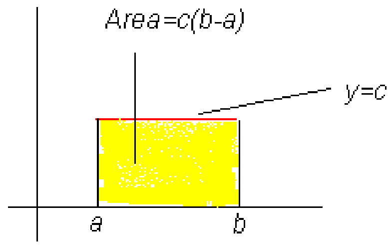

Figure 1.5

Property 2 says that the integral of a sum is the sum of the integrals. The area under $f+g$ is the area under f plus the area under g. This property follows from the property of limits and sums.
Property 3 tells us that a constant (but only a constant) can be taken in front of an integral sign. This also follows from the properties of limits and sums.
Property 4 follows from property 2 and 3, using $c=-1$.
Property 5 tells us we can find the area under the graph between a and c, by splitting it up into two areas, between a and b, and between b and c.

We now find our rule for evaluating the definite integral:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where the derivative of $F(x)$ is $f(x)$, i.e. F is any anti-derivative of f.

For example, if we differentiate $F(x) = \frac{x^3}{3}$ we obtain $f(x) = x^2$, so F(x) is an anti-derivative of $f(x)$.

Thus $\int_0^1 x^2 dx = F(1) - F(0) = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$.
Therefore the area under the curve $f(x) = x^2$ between 0 and 1, is equal to a third, or 0.33 recurring. Incidentally this answers our previous question which we attempted using the sum of the areas of n rectangles.

The fundamental theorem of calculus motivates this use of the evaluation theorem. In short, it states that differentiation and integration are opposite processes. Thus, if we start with a function $F(x)$, and differentiate it to obtain $f(x)$, [i.e. $F'(x) = f(x)$], if we then integrate the function $f(x)$, the result will be the initial function $F(x)$. Similarly if we integrate $f(x)$ to obtain $F(x)$, [i.e. $\int f(x)dx = F(x) + C$ where C is an arbitrary constant[1]]. Thus to find the integral of a function $f(x)$, we must find the function which when differentiated yields $f(x)$. This theorem is very useful to us, as otherwise whenever we wish to find the value of the area that lies underneath a curve, we have to go through the entire process of finding the limit of the sum of the areas of n approximating rectangles, which is a time consuming process! Prior to the discovery of the fundamental theorem, finding areas, volumes and other similar types of problems were nigh on impossible.

For completeness, the fundamental theorem is presented below:

[1] More about the arbitrary constant a little later

The fundamental theorem of calculus:

Suppose *f(x)* is a continuous function on the closed interval [*a,b*]

1. If $g(x) = \int_a^x f(t)dt$ then $g'(x) = f(x)$ i.e. $g'(x) = \frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$

2. $\int_a^b f(x)dx = F(b) - F(a)$

where the derivative of *F(x)* is *f(x)*, i.e. F is any anti-derivative of *f*. What it says, roughly speaking, is that if you integrate a function and then differentiate the result, you retrieve the original function.

We now need to discuss two different types of integrals – definite and indefinite. A definite integral involves finding the integral of a function between two number limits i.e. $\int_a^b f(x)dx$. The answer to a definite integral is a number, as we know according to the evaluation rule the answer to this is just the anti-derivative *F(x)* evaluated between a and b, i.e. *F(b)-F(a).* An indefinite integral yields a function of *x* as its answer (if we are integrating with respect to *x*). An indefinite integral is an integral of the form $\int f(x)dx$ (i.e without upper and lower limits) and the solution is

$$\int f(x)dx = F(x) + C$$

where C is an arbitrary constant which can take on any value.

The reason we include the arbitrary constant is illustrated in the following example.

Given the problem:

$\int 3x^2 dx$ a potential solution is x^3 as this is an antiderivative of the cubic function (If we differentiate x^3 we obtain $3x^2$. However $x^3 + 4$ is also a solution to this problem, as is $x^3 - 100$. This is because when differentiating these expressions, the constant differentiated moves to zero. So it would appear that the most general form to give the answer to this problem would be as follows:

$\int 3x^2 dx = x^3 + C$,

where C is an arbitrary constant.

Just a small note on arbitrary constants – when we add two together, we obtain a third one which has aggregated the first two, when multiplying, dividing adding or subtracting a number by/to/from an arbitrary constant, the result is just the arbitrary constant. However the arbitrary constant when multiplied by a function of *x*, will stay as just that:

$\int f(x) + g(x)dx = F(x) + C_1 + G(x) + C_2 = F(x) + G(x) + C$

where C_1 and C_2 are two arbitrary constants, and *F(x)* and *G(x)* are two anti-derivates of *f(x)* and *g(x)* respectively.

$3\int f(x)dx = 3[F(x) + C] = 3F(x) + 3C = 3F(x) + C$

However:

$$x\int f(x)dx = x[F(x)+C] = xF(x)+Cx$$

We now turn to some rules of integration (definite and indefinite) and then some examples.

1. $\int cf(x)dx = c\int f(x)dx$

2. $\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$

3. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$ (n cannot equal –1)

4. $\int \dfrac{1}{x}dx = \ln|x| + C$

5. $\int e^x dx = e^x + C$

6. $\int a^x dx = \dfrac{a^x}{\ln a} + C$

7. $\int \sin x dx = -\cos x + C$

8. $\int \cos x dx = \sin x + C$

Remember – to check the answer to any integration sum just differentiate it and you should arrive back at the original function.

Some examples:

1. $\int 3dx = 3x + C$

2. $\int dx = x + C$

3. $\int \sqrt{x^3}dx = \int x^{\frac{3}{2}}dx = \dfrac{2}{5}x^{\frac{5}{2}} + C$

4. $\int_1^2 \dfrac{1}{x^4}dx = \int_1^2 x^{-4}dx = \left.\dfrac{x^{-3}}{-3}\right|_1^2 = -\dfrac{1}{24} - (-\dfrac{1}{3}) = \dfrac{7}{24}$

5. $\int [10x^4 + \sin x]dx = \dfrac{10x^5}{5} - \cos x + C = 2x^5 - \cos x + C$

6. $\int_0^3 (x^3 - 6x)dx = \left.\dfrac{x^4}{4} - 6\dfrac{x^2}{2}\right|_0^3 = \left.\dfrac{x^4}{4} - 3x^2\right|_0^3 = \dfrac{81}{4} - 3(9) - 0 + 0 = -6.75$

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t}dt = \int_1^9 (2t + t\sqrt{t} - t^{-1})dt = \int_1^9 (2t + t^{\frac{3}{2}} - t^{-1})dt = \left.t^2 + \frac{2}{5}t^{\frac{5}{2}} - \ln t\right|_1^9$$

7. $= 81 + \dfrac{2}{5}(243) - \ln 9 - 1 - \dfrac{2}{5} + \ln 1$

$= 174.603$

8. $\int_1^4 (t^2 - t - 6)dt = \frac{t^3}{3} - \frac{t^2}{2} - 6t \Big|_1^4 = -\frac{9}{2}$

A few more useful properties:

$$1. \int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$2. \int_a^a f(x)dx = 0$$

We are going to be looking at two very useful techniques used in integration: use of substitution, and integration by parts. There are a whole host of other techniques which can be useful, however it is these two which are most useful to us in economics.

Integration using Substitution

We use substitution, when the integrand contains a function and its own derivative. i.e:

$$\int f'(x)f(x)dx$$

For example:

$$\int (x^3 + 5x^2 - 10)(3x^2 + 10x)dx$$

If this is the case, we can make use of the following substitution: We let *u* equal to the function whose derivative we can spot (or create, using a constant: more about this later).

Let u = $x^3 + 5x^2 - 10$
Then we know that
$du = (3x^2 + 10x)dx$
When we then substitute the values of u and du into the integral, we obtain the following integral:
$\int udu$
which has the answer
$\int udu = \frac{u^2}{2} + C$
but: u = $x^3 + 5x^2 - 10$

Therefore our final answer is

$$\int (x^3 + 5x^2 - 10)(3x^2 + 10x)dx = \frac{(x^3 + 5x^2 - 10)^2}{2} + C$$

Thus the general rule solution for the problem is as follows:

$\int f'(x)f(x)dx = \frac{[f(x)]^2}{2} + C$ or more simply $\int (f'.f)dx = \frac{f^2}{2} + C$

However the substitution rule can be used for more complicated examples, when our function *f(x)* whose derivative we can spot occurs inside another function:
For example:

$$\int f'g(f)dx \qquad \text{or} \qquad \int (6x+18)\sqrt{(3x^2+18x-23)}dx$$

In these cases the procedure does not change at all – we still make the substitution as follows:
Let $u = f(x)$
Therefore $du=f'(x)dx$
And proceed as usual:

Some examples:

1. $\int (6x+18)\sqrt{(3x^2+18x-23)}dx$

Let $u = (3x^2+18x-23)$

Therefore $du = (6x+18)dx$

Therefore our transformed integral is as follows:

$$\int \sqrt{u}du = \int u^{\frac{1}{2}}du = \frac{2u^{\frac{3}{2}}}{3} + C = \frac{2(3x^2+18x-23)^{\frac{3}{2}}}{3} + C$$

2. Sometimes we can find a function whose derivative we can create as follows: However only if we introduce a constant function.

For example:

$\int e^{(4x+5)}dx$

If we let u = (4x+5), we know $du = 4dx$. However while we have the dx, we do not have a 4. This is easily solved however through the following manipulation:

$\frac{1}{4}\int 4e^{(4x+5)}dx$

This integral now contains a function and its derivative, thus substitution can be used:
Therefore let u = (4x+5), and du = 4dx. The integral becomes:

$$\frac{1}{4}\int e^u du = \frac{1}{4}e^u + C = \frac{1}{4}e^{(4x+5)} + C$$

Remember, the substitution of u into this function is a device that we employ. Therefore our final answer ought not to contain u, as the original problem does not contain it. Always remember to substitute back for u.

Note: Unlike differentiation, there exists no general formula giving the integral of a product of two function i.t.o. the separate integrals of those functions. There is also no general formula giving the integral of a quotient of 2 functions in terms of their separate integrals. As a result, integration is trickier than differentiation, on the whole.

Integration by Parts

We use this technique when we have to integrate a product:

$$\text{E.g.} \quad \int f(x).g(x)dx$$

When we are given this type of example, we make use of the following formula:

$$\boxed{\int f(x).g'(x)dx = f(x).g(x) - \int f'(x).g(x)dx}$$

For example:

$$\int xe^x dx$$

We pick f as the function which is easy to differentiate, and g' as the function which is easy to integrate. Often picking a squared or cubic term for your f is a good idea, as f' will have a power that is then one lower, and hence simpler. It is a very good idea to make yourself a mini table with f', g', f and g, to keep things straight. Also, note that when finding g, we do not bother with the arbitrary constant.

$$\int xe^x dx$$

Therefore:

$$\begin{array}{ll} f = x & g' = e^x \\ f' = 1 & g = e^x \end{array}$$

$$\begin{aligned} \int xe^x dx &= xe^x - \int (1)e^x dx \\ &= xe^x - e^x + C \end{aligned}$$

Thus we have managed to use the formula to integrate our original question.

(Can also use the alternative notation used in Chiang: where $\int vdu = uv - \int udv$)

Another example:

$$\int xsinxdx$$

Therefore:

$$\begin{array}{ll} f = x & g' = \sin x \\ f' = 1 & g = -\cos x \end{array}$$

$$\begin{aligned} \int xe^x dx &= -x\cos x + \int \cos x dx \\ &= -x\cos x + \sin x + C \end{aligned}$$

An example with a trick:

$$\int \ln x dx$$

Obviously we do not know the integral of $\ln x$, that is why we are using this method. So we make $\ln x$ equal to f, and we can then find its derivative. But then what will our g' be? Simple, make it 1.

$$\begin{array}{ll} f = \ln x & g' = 1 \\ f' = \frac{1}{x} & g = x \end{array}$$

$$\begin{aligned} \int \ln x dx &= x \ln x - \int x \left(\frac{1}{x}\right) dx \\ &= x \ln x - x + C \end{aligned}$$

This is a handy trick which can also be used to find the integrals of some of the trigonometric functions.

Some more examples:

$$\int xe^{2x} dx$$

Therefore:

$$\begin{array}{ll} f = x & g' = e^{2x} \\ f' = 1 & g = \frac{e^{2x}}{2} \end{array}$$

$$\begin{aligned} \int xe^{2x} dx &= \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2} dx \\ &= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C \end{aligned}$$

Which can then be re-written if we want to:

$$\int xe^{2x} dx = \frac{e^{2x}}{2} \left(x - \frac{1}{2}\right) + C$$

Another example:

$$\int x \ln x dx$$

$$\begin{array}{ll} f = \ln x & g' = x \\ f' = \frac{1}{x} & g = \frac{x^2}{2} \end{array}$$

$$
\begin{aligned}
\int x \ln x dx &= \frac{x^2}{2} \ln x - \int \left(\frac{1}{x}\right) \frac{x^2}{2} dx \\
&= \frac{x^2}{2} \ln x - \int \frac{x}{2} dx \\
&= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C
\end{aligned}
$$

Another example:

$$\int (\ln x)^2 \, dx$$

$$
\begin{array}{ll}
f = (\ln x)^2 & g' = 1 \\
f' = 2(\ln x) \frac{1}{x} & g = x
\end{array}
$$

$$
\begin{aligned}
\int (\ln x)^2 \, dx &= x(\ln x)^2 - \int x.2 \ln x \left(\frac{1}{x}\right) dx \\
&= x(\ln x)^2 - \int 2 \ln x dx \\
&= x(\ln x)^2 - 2(x \ln x - x) + C
\end{aligned}
$$

The last line uses a result that we proved a few examples ago.

Now let's try a definite integral using integration by parts:

$$\int_0^1 te^{-t} dt$$

$$
\begin{array}{ll}
f = t & g' = e^{-t} \\
f' = 1 & g = -e^{-t}
\end{array}
$$

We first calculate the indefinite integral, then go back and substitute in the limits.

$$
\begin{aligned}
\int te^{-t} dt &= -te^{-t} + \int e^{-t} dt \\
&= -te^{-t} - e^{-t} + C
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\int_0^1 te^{-t} dt &= \left[-te^{-t} - e^{-t}\right]\Big|_0^1 \\
&= \left[-1e^{-1} - e^{-1}\right] - \left[-e^{-0}\right] \\
&= 1 - 2e^{-1} \\
&= 1 - \frac{2}{e}
\end{aligned}
$$

You should now make sure you can do the integration practice questions.

An example of an economic application of integrals:

One simple application is to find a 'total' quantity from a 'marginal' quantity. Suppose a firm has a marginal cost $C'(x) = 1 + 2e^{\frac{x}{3}}$ where x denotes output. Then total cost is:

$$\begin{aligned} C(x) &= \int \left(1 + 2e^{\frac{x}{3}}\right) dx \\ &= x + 6e^{\frac{x}{3}} + B, \end{aligned}$$

where B is the constant of integration.

Differential Equations

A DE is an equation containing a function $y = f(x)$ and one or more of its derivatives, i.e. y', y'' etc. It shows the relationship between the function and its derivatives. If only the first derivative $\frac{dy}{dt}$ is present, the differential equation is said to be of the first order.

Some examples include:

$$y' = y'' + y$$

$$\frac{dy}{dx} + xy = x$$

$$3y - 2y' = x$$

As you can see, these equations contain a dependent variable y, some or more of its derivatives y', y'' and some occurrences of the independent variable, in these cases x. Our aim with differential equations is to solve for our function y, in terms of the independent variable x, with none of the derivatives still present. How we do this depends largely on the type of differential equation present. What form a DE takes will determine how we solve it. Your main task in this section is to identify the type of DE. Once this is done it is usually fairly simple to proceed from there.

We will be considering 4 types of differential equations:

1. Directly integrable DEs
2. Separable DEs
3. 1st Order Linear DEs
4. 2nd Order DEs

1 Directly Integrable Differential Equations:

This type of DE contains only one of the function's derivatives, and a function of the independent variable. For example:

1. $y' = x^2 + 2$
2. $\frac{d^2y}{dx^2} + x = e^x$
3. $y''' = 6$

Solving these DE's just requires that we integrate as many times as is necessary, including the necessary constants of integration. Eg:

1. $y' = x^2 + 2$
   $y = \frac{x^3}{3} + 2x + C$
2. $\frac{d^2y}{dx^2} + x = e^x$
   $\frac{d^2y}{dx^2} = e^x - x$
   $\frac{dy}{dx} = e^x - \frac{x^2}{2} + C$
   $y = e^x - \frac{x^3}{6} + Cx + D$
3. $y''' = 6$
   $y'' = 6x + C$
   $y' = 3x^2 + Cx + D$
   $y = x^3 + \frac{Cx^2}{2} + Dx + E$

This procedure gives us the *general* solution for these DE's. To obtain the specific solution, i.e. without the presence of arbitrary constants, we require some **initial conditions**.

Eg: $y(0) = 2$ or $y'(0) = 1$

To solve for the specific solution to one of these DE's, we use the initial conditions as follows:

$y''' = 6$

$y(0) = 1$

$y'(0) = 2$

We know that the general solution to this problem is:

$y' = 6x + C$

$y = 3x^2 + Cx + D$

Therefore we substitute in the initial conditions, obtain some simple simultaneous equations, and solve for the two arbitrary constants.

$y = 3x^2 + Cx + D$

$y(0) = 1 = D$

$y'(0) = 2 = C$

$y = 3x^2 + 2x + 1$

Directly integrable DE's are the simplest type of DE, and sadly we won't encounter them very often.

2 Separable DE's

The general form for a separable DE is:

$$\boxed{\frac{dy}{dx} = f(x).g(y)}$$

i.e. the RHS can be separated into a function of x times a function of y. Sometimes you may have to do some manipulation to achieve this.

E.g.

$$\begin{aligned}(1+x)dy - ydx &= 0\\ (1+x)dy &= ydx\end{aligned}$$

becomes

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{1+x}\\ &= y.\frac{1}{1+x}\end{aligned}$$

which is separable.

E.g.

$$xy^4dx + \left(y^2+2\right)e^{-3x}dy = 0$$

becomes

$$\frac{dy}{dx} = e^{3x}.\frac{-xy^4}{y^2+2}$$

which is separable.

To solve a separable DE, we rewrite it as follows, and then integrate both sides with respect to x and y.

$$\boxed{\begin{aligned}\frac{dy}{dx} &= f(x).g(y)\\ \frac{1}{g(y)}dy &= f(x)dx\\ \int\frac{1}{g(y)}dy &= \int f(x)dx\end{aligned}}$$

Since the LHS involves only y and the RHS involves only x, the solution method is known as separating the variables. If possible, one then solves for y in terms of x, but if not, one simply leaves the equation in simplest form, making sure the derivatives are no longer present.

For example:

1. $\frac{dy}{dx} = \frac{y}{1+x}$
   $\frac{1}{y}dy = \frac{1}{1+x}dx$
   $\int\frac{1}{y}dy = \int\frac{1}{1+x}dx$
   $\ln y = \ln(1+x) + C$
   $e^{\ln y} = e^{\ln(1+x)+C}$

$y = A(1+x)$,
where
$A = e^C$

2. $\frac{dy}{dx} = \frac{-x}{y}$
   $ydy = -xdx$
   $\int ydy = \int -xdx$
   $\frac{y^2}{2} = -\frac{x^2}{2} + C$

It is often hard or almost impossible to solve explicitly for y in terms of x in a separable DE. We then usually are content to just solve the DE, i.e. eliminate the presence of derivatives. You will waste large portions of time in your exam if you do not realise this.

E.g.

$$\begin{aligned} \frac{dy}{dt} &= \frac{t^3+1}{y^6+1} \\ \int \left(y^6+1\right) dy &= \int \left(t^3+1\right) dx \\ \frac{y^7}{7} + y &= \frac{t^4}{4} + t + C \end{aligned}$$

It is impossible here to solve for y in terms of t, so we just leave it as is.

An initial value problem:

$\frac{dy}{dx} = y - 4$

$y(7) = 5$

$\int \frac{1}{y-4} dy = \int dx$

$\ln(y-4) = x + C$

Sub in $y(7) = 5$

$\ln(5-4) = 7 + C$

$\ln 1 = 7 + C$

$C = -7$

$\ln(y-4) = x - 7$

$y - 4 = e^{x-7}$

$y = 4 + e^{x-7}$

Some more examples:

1. $dx + e^{3x} dy = 0$
   $dy = -e^{-3x} dx$
   $\int dy = \int -e^{-3x} dx$
   $y = \frac{e^{-3x}}{3} + C$

2. $(x+1)\frac{dy}{dx} = x+6$
   $\int dy = \int \frac{x+6}{x+1} dx$
   $\int dy = \int \frac{x+1+5}{x+1} dx$
   $\int dy = \int 1 + \frac{5}{x+1} dx$
   $y = x + 5\ln(x+1) + C$

The next example uses *integration by parts*:

$e^x \frac{dy}{dx} = 2x$

$\int dy = \int 2xe^{-x} dx$

For $\int 2xe^{-x} dx$:

$f = 2x$

$f' = 2$

$g = e^{-x}$

$g' = -e^{-x}$

$\int 2xe^{-x} dx = -2xe^{-x} - 2e^{-x} + C$

$y = -2xe^{-x} - 2e^{-x} + C$

Solve the differential equation $\frac{dy}{dt} + ay = 0$, where a is non-zero constant.

Separating the variable gives $\int y^{-1} dy = \int (-a) dt$.

Integrating we obtain :

$\ln y = -at + C$, where C is an arbitrary constant.

Hence

$y = e^{C-at} = e^C e^{-at}$

Setting $A = e^C$, we may write the general solution as $y = Ae^{-at}$, where A is a constant.

The next type of differential equation we encounter:

3 First Order Linear Differential Equations:

The general form of a first order linear DE is:

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

NB: If your equation is NOT in this format, you must REWRITE it before you can work with it. The general method for a linear first order DE is as follows:

1. Put it into standard form.
$$\frac{dy}{dx} + P(x)y = Q(x)$$
2. Identify $P(x)$ and find the integrating factor I, which is equal to:
$$I = e^{\int P(x) dx}$$

3. Multiply LHS and RHS by I.

$$e^{\int P(x)dx}\frac{dy}{dx}+e^{\int P(x)dx}P(x)y=e^{\int P(x)dx}Q(x)$$

4. The LHS becomes the derivative of the product of I and y, so we rewrite it as such. We are essentially doing the product rule for differentiation here, but in REVERSE. Instead of going from $(f.g)'=f'g+fg'$, we are moving from the RHS to the LHS. Thus the LHS of the equation in step three becomes:

$$\frac{d}{dx}(I.y)=Q(x).I$$

5. We then integrate both sides. For the LHS, this is simple, as the integral and derivative signs cancel each other out, and we simply obtain $I.y$ on that side.

$$\begin{aligned}\int\frac{d}{dx}(I.y) &= \int[Q(x).I]\,dx\\ I.y &= \int[Q(x).I]\,dx\end{aligned}$$

6. The last step is simply to solve for y. Like the separable equations, it is sometimes not possible to solve explicitly for y in terms of x, and we just work towards eliminating derivatives.

Example:

Given:

$$x.\frac{dy}{dx}-4y=x^6e^x$$

1. Rewrite in standard form.

$$\frac{dy}{dx}-\frac{4}{x}y=x^5e^x$$

2. Find the integrating factor.

$$P(x) = -\frac{4}{x}$$

$$\begin{aligned}I &= e^{\int-\frac{4}{x}dx}\\ &= e^{-4\ln x}\\ &= e^{\ln x^{-4}}\\ &= x^{-4}\end{aligned}$$

3. Multiply LHS and RHS by it.

$$\begin{aligned}x^{-4}\frac{dy}{dx}-x^{-4}\frac{4}{x}y &= x^{-4}x^5e^x\\ x^{-4}\frac{dy}{dx}-4x^{-5}y &= xe^x\end{aligned}$$

4. Rewrite the LHS as $\frac{d}{dx}(I.y)$.

$$\frac{d}{dx}\left(x^{-4}y\right)=xe^x$$

5. Integrate both sides.

$$\begin{aligned}\int \frac{d}{dx}\left(x^{-4}y\right) &= \int xe^x dx \\ x^{-4}y &= xe^x - e^x + C\end{aligned}$$

6. Solve for y (which is possible in this case).

$$y = x^5e^x - x^4e^x + Cx^4$$

Another example:

(Note: although it is a separable DE but we solve it using the method for first-order linear DE's).

$$\begin{aligned}(x^2+9)\frac{dy}{dx} + xy &= 0 \\ \frac{dy}{dx} + \frac{x}{(x^2+9)}y &= 0\end{aligned}$$

$$\begin{aligned}I &= e^{\int \frac{x}{(x^2+9)}dx} \\ &= e^{\frac{1}{2}\int \frac{2x}{(x^2+9)}dx} \\ &= e^{\frac{1}{2}\ln(x^2+9)} \\ &= e^{\ln\sqrt{(x^2+9)}} \\ &= \sqrt{x^2+9}\end{aligned}$$

Now multiply the LHS and RHS by the integrating constant:

$$\begin{aligned}\sqrt{x^2+9}\frac{dy}{dx} + \sqrt{x^2+9}\frac{x}{(x^2+9)}y &= 0 \\ \frac{d}{dx}\left(\sqrt{x^2+9}.y\right) &= 0 \\ \int \frac{d}{dx}\left(\sqrt{x^2+9}.y\right) &= \int 0dx \\ \sqrt{x^2+9}.y &= C \\ y &= \frac{C}{\sqrt{x^2+9}}\end{aligned}$$

Another example, using an initial value:

Solve $x\frac{dy}{dx} + y = 2x$, where $y(1) = 0$.

Solution:

$$\frac{dy}{dx} + \frac{1}{x}y = 2$$

Example 2:

$$\begin{aligned} 3\frac{dy}{dx} + 12y &= 9 \\ \frac{dy}{dx} + 4y &= 3 \end{aligned}$$

$$\begin{aligned} I &= e^{\int 4dx} \\ &= e^{4x} \end{aligned}$$

$$\begin{aligned} e^{4x}\frac{dy}{dx} + 4e^{4x}y &= 3e^{4x} \\ \frac{d}{dx}\left[e^{4x}.y\right] &= 3e^{4x} \\ \int \frac{d}{dx}\left[e^{4x}.y\right] &= \int 3e^{4x}dx \\ e^{4x}y &= \frac{3e^{4x}}{4} + C \\ y &= \frac{3}{4} + Ce^{-4x} \end{aligned}$$

Homogeneous and Non-Homogeneous Cases of Differential Equations:

The equation $\frac{dy}{dx} + ay = 0$, where a is some constant, is said to be homogenoeus on account of the zero constant term. This equation can be re-written as

$$\frac{1}{y}\frac{dy}{dt} = -a$$

The solutions to this DE are as follows:

$$\begin{aligned} y(t) &= Ae^{-at} && \text{[general solution]} \\ y(t) &= y(0)e^{-at} && \text{[definite solution]} \end{aligned}$$

Non-Homogenous Case:

When a constant takes the place of the zero in the above equations, we have a **non-homogeneous** linear differential equation:

$$\frac{dy}{dt} + ay = b$$

The solution to this equation consists of the sum of two terms, one of which is called the *complementary* function (denoted by y_c) and the other known as the *particular integral*, denoted by y_p.

In this case:

$$
\begin{aligned}
y_c = &\quad Ae^{-at} && \text{[general solution]} \\
y_p = &\quad \tfrac{b}{a}, a \neq 0 && \text{[definite solution]}
\end{aligned}
$$

General solution of the complete equation is the sum of y_c and y_p:

$$
\begin{aligned}
y(t) &= y_c + y_p \\
&= Ae^{-at} + \frac{b}{a}
\end{aligned}
$$

Try the following exercise and see if you get the same answer:

$$
\begin{aligned}
xy' + y &= e^x \\
y(1) &= 2
\end{aligned}
$$

The solution is:

$$x.y = e^x(x-1) + 2$$

Our last type of problem involves:

4 2nd Order Differential Equations

General Form:

$$\boxed{y'' + a_1 y' + a_2 y = b}$$

Where a_1, a_2 and b are constants.

2nd order DE's can involve cases wherea_1, a_2 and b are not constant but for our purposes we don't need to worry about these.

A general solution for the DE above is:

$$\boxed{y = y_c + y_p}$$

where y_c denotes the *complementary function* and y_p denotes the *particular integral.*

We find y_c and y_p using the following set of rules:

In solving for the particular integral, we can distinguish between 3 different cases:

Case 1: If a_2 is non-zero

$$y_p = \frac{b}{a_2}$$

Case 2: If a2 = zero

$$y_p = \frac{b}{a_1}t$$

Case 3: Both a_2 and a_1 are zero

$$y_p = \frac{b}{2}t^2$$

And to find y_c , we find the solution to the homogeneous version of the DE, i.e. y_c is the solution to:

$$y'' + a_1y' + a_2y = 0$$

Which is the same as saying that $b = 0$.

The complementary solution to the second-order differential equation takes the form:

$$y(t) = A_1e^{r_1t} + A_2e^{r_2t}$$

where A_1, A_2 are two arbitrary constants and r_1 and r_2 are the roots from the characteristic equation:

$$r^2 + ar + b = 0$$

given by

$$r_1, r_2 = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4ab}\right)$$

In other words:

To solve this, we form the corresponding characteristic equation:

$$r^2 + a_1r + a_2 = 0$$

And solve for r_1 and r_2, which are the roots of the characteristic equation.

These then yield the following general solutions for y_c:

In the case r_1 and r_2, are unequal:

$$\begin{aligned} r_1 &\neq r_2 \\ y_c &= Ae^{r_1t} + Be^{r_2t} \end{aligned}$$

In the case where r_1 and r_2, are equal:

$$\begin{aligned} r_1 &= r_2 \\ y_c &= Ae^{rt} + Bte^{rt} \end{aligned}$$

There is also the case where r_1 and r_2, are complex roots, which we ignore for our purposes. Hopefully you all have not forgotten the formula to solve a quadratic equation, but in case you have:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

So our final solution is:

$$y = y_c + y_p$$

Where we have found y_c and y_p using the relevant rules.

Some examples:

$$y'' + y' - 2y = -10$$

Therefore:

$$a_1 = 1 \qquad a_2 = -2 \qquad b = -10$$

We find the appropriate case for y_p, which is the first one, i.e. a_2 is non-zero.

$$y_p = \frac{b}{a_2} = \frac{-10}{-2} = 5$$

We then find the solution for y_c which is the solution to:

$$\begin{aligned} y'' + y' - 2y &= 0 \\ r^2 + r - 2 &= 0 \\ (r+2)(r-1) &= 0 \end{aligned}$$

$$r_1 = -2 \qquad r_2 = 1$$

This is the case of unequal roots:

$$r_1 \neq r_2$$

$$\begin{aligned} y_c &= Ae^{r_1 t} + Be^{r_2 t} \\ &= Ae^{-2t} + Be^{t} \\ y &= y_c + y_p \\ y_p &= 5 \\ y &= 5 + Ae^{-2t} + Be^{t} \end{aligned}$$

Given initial conditions, we could also solve for A and B.

For example, given that $y(0) = 12, y'(0) = -2$

We can now solve for the specific solution:

$$\begin{aligned} y &= 5 + Ae^{-2t} + Be^{t} \\ 12 &= 5 + A + B \\ 7 &= A + B \end{aligned}$$

$$\begin{aligned} y' &= -2Ae^{-2t} + Be^{t} \\ -2 &= -2A + B \end{aligned}$$

$$\begin{aligned} 7 &= A + B \\ -2 &= -2A + B \\ -9 &= -3A \end{aligned}$$

$$\begin{aligned} A &= 3 \\ B &= 4 \end{aligned}$$

$$y = 5 + 3e^{-2t} + 4e^{t}$$

Example Two: (This is a case of repeated roots)

$$y'' + 6y' + 9y = 27$$

$$y(0) = 5 \qquad y'(0) = -5$$

$$a_1 = 1 \qquad a_2 = 6 \qquad b = 27$$

$$y_p = \frac{27}{9} = 3$$

$$\begin{aligned} y'' + 6y' + 9y &= 0 \\ r^2 + 6r + 9 &= 0 \\ (r+3)(r+3) &= 0 \\ r &= -3 \end{aligned}$$

$$r_1 = r_2$$

$$\begin{aligned} y_c &= Ae^{rt} + Bte^{rt} \\ &= Ae^{-3t} + Bte^{-3t} \\ y_p &= 3 \end{aligned}$$

$$\begin{aligned} y(t) &= y_c + y_p \\ &= Ae^{-3t} + Bte^{-3t} + 3 \\ y'(t) &= -3Ae^{-3t} + Be^{-3t} - 3Bte^{-3t} \end{aligned}$$

$$y(0) = 5$$

$$
\begin{aligned}
5 &= 3 + A \\
A &= 2 \\
y'(0) &= -5 \\
-5 &= -3A + B \\
B &= 1
\end{aligned}
$$

$$y = 3 + 2e^{-3t} + te^{-3t}$$

DIFFERENCE EQUATIONS

What are difference equations?

Whereas differential equations deal with problems in continuous time, difference equations are concerned with problems in discrete time. Here the variable *t* is allowed to take integer values only, making the concept of a derivative or differential no longer appropriate.
Instead the pattern of change of the variable *y* must be described by 'differences', rather than by derivatives or differentials of *y(t).*

When we are dealing with discrete time, the value of variable *y* will change only when the variable *t* changes from one integer value to the next, such as from *t* =1 to *t*=2. Meanwhile nothing is supposed to happen to *y*. It is therefore more convenient to interpret the values of *t* as referring to *periods* – rather than *points* – of time, with *t* =1 denoting period 1, *t*=2 denoting period 2 and so forth. Then we can regard *y* as having one unique value in each time period.
In view of this interpretation, the discrete-time version of economic dynamics is often referred to as *period analysis.*

In difference equations, the pattern of change is represented by the difference quotient $\frac{\Delta y}{\Delta t}$. *t* can only take integer values, so if we compare the values of *y* in two consecutive periods, we must have *t* = 1. For this reason the difference quotient $\frac{\Delta y}{\Delta t}$ can be simplified to the expression Δy ; this is called the *first-difference* of *y.*
This refers to the rate of change of *y* in period *t*, which is equal to $y_{t+1} - y_t$.

We can therefore define the forward difference operator Δ by $\Delta y_t = y_{t+1} - y_t$.
Application of the operator Δ may be regarded as the discrete-time counterpart of differentiation with respect to time.

Difference equations are therefore like differential equations, except that derivatives are replaced by differences. This the discrete-time analogue of the differential equation $\frac{dy}{dt} + ay = b$, where *a* and *b* are constants, is the first-order difference equation

$$\Delta y_t + ay_t = b.$$

Recalling the definition of the operator Δ, we may write this equation as:

$y_{t+1} + cy_t = b$, where *c*=-1.
Since this equation simply relates the value of *y* in a period to its value in the previous period, it can be written in the equivalent form:

$$y_t + cy_{t-1} = b$$

Solving first-order difference equations.

In this section we are concerned with difference equations which contain y_t and y_{t+1}, but not y_{t+2} or further terms in the sequence.

Before looking at a general method to solve difference equations, we will consider an iterative method, which will additionally provide some insight into the nature of the solution to a difference equation. Particularly in the case of first order difference equations, simple iteration of the differencing or recursive rule 'plays out' the recursive rule over a number of time periods, in order to see whether we can depict a general characterisation of the time path in *y(t).*

For example:

$$y_{t+1} = y_t + 2,\ y_0 = 15.$$

From this equation we can deduce step-by-step that:

$$y_1 = y_0 + 2$$

$$y_2 = y_1 + 2 = (y_0 + 2) + 2 = y_0 + 2(2)$$

$$y_3 = y_2 + 2 = [y_0 + 2(2)] + 2 = y_0 + 2(2)$$

....

And in general, for any period *t*

$$y_t = y_0 + t(2) = 15 + 2t$$

This equation indicates the *y* value of any time period and therefore constitutes the solution of the difference equation.

This method is crude and quickly runs into limitations and essentially corresponds to the process of direct integration, which is feasible for certain types of differential equations. For this reason we have to use more general solution methods.

General Solutions

To find the general solution of the difference equation means finding a formula giving all sequences { y_t } which satisfy $y_t + cy_{t-1} = b$. As in the differential equation case this will contain an arbitrary constant which will be tied down if we specify the value of y_t for some particular *t.*

We start with the simple case where $b = 0$. Then $y_{t+1} = -cy_t$ for all *t* and hence:

$$y_1 = -cy_0$$

$$y_2 = -cy_1 = (-c)^2 y_0$$

$$y_3 = (-c)^3 y_0$$

and so on. It follows that:

$$y_t = (-c)^t y_0 \text{ for } t = 0,1,2\ldots.$$

To summarise, if $b = 0$, (i.e. a linear, homogenous difference equation) the general solution to $y_{t+1} + cy_t = b$ is $y_t = A(-c)^t$, where A is an arbitrary constant which may be interpreted as y_0.

In the case where $b \neq 0$, (a liner, non-homogenous DE) we use a method similar to the one we used to solve the analogous differential equation.

The general solution will consist of the sum of two components: a *particular* solution y_p, which is any of the complete non-homogenous equation $y_t + cy_{t-1} = b$, and a *complementary* function y_c, which is the general solution of the reduced equation $y_t + cy_{t-1} = 0$.

First we consider the *complementary function*:

Our experience with the example above suggests that we might try a solution of the form $y_t = A(-c)^t$.

Now we must look for the *particular solution*, which relates to the complete equation $y_{t+1} + cy_t = b$.

We note that for y_p we can choose any solution of $y_{t+1} + cy_t = b$. Therefore, if a trial solution of the simplest form $y_t = k$ (a constant) can work out, no real difficulty will be encountered.
Now, if $y_t = k$, then y will maintain the same constant value over time and we must have $y_{t+1} = k$. If we substitute these values into $y_{t+1} + cy_t = b$, we get:

$$k + ck = b \text{ and } k = \frac{b}{1+c}.$$

Since this particular value of k satisfies the equation, the particular solution can be written as:

$$y_p (= k) = \frac{b}{1+c} \quad (c \neq -1).$$

Since this is a constant, a stationary equilibrium is indicated in this case.

What if $c = -1$?
If $c = -1$, the particular solution $y_p = \dfrac{b}{1+c}$ is not defined and some other solution of the non-homogenous equation $y_{t+1} + cy_t = b$ must be found.
In this case we use the trick of trying a solution of the form $y_t = kt$. This implies that $y_{t+1} = k(t+1)$.
Substituting these into $y_{t+1} + cy_t = b$ we find:

$$k(t+1) + ckt = b$$
$$and$$
$$k = \frac{b}{t+1+ct} = b \text{ [becasue } c = -1]$$
$$\therefore y_p (= kt) = bt$$

This form of the particular solution is a non-constant function of *t,* it therefore represents a moving equilibrium.

Adding y_c and y_p together, we may now write the general solution in one of the two following forms:

$y_t = A(-c)^t \; + \dfrac{b}{1+c}$ [general solution, case of $c \neq -1$]

$y_t = A(-c)^t \; + bt = A + bt$ [general solution, case where $c = -1$].

To eliminate the arbitrary constant., we have to use the initial condition that $y_t = y_0$ when $t = 0$.
Letting $t = 0$, we have:
For $c = -1$:

$y_t = A + \dfrac{b}{1+c}$ and $A = y_0 - \dfrac{b}{1+c}$

$\therefore y_t = (y_0 - \dfrac{b}{1+c})(-c)^t + \dfrac{b}{1+c}$

and for $c \neq -1$

$y_0 = A$, so the definite version of this equation is:

$y_t = y_0 + ct$

Example:

Find the solution of the difference equation $y_{t+1} - \dfrac{1}{2} y_t = 2$, which satisfies the boundary condition $y_0 = 2$.

As a particular solution y_p we try $y_t = k$ (a constant).
This satisfies the equation provided :

$k - \dfrac{1}{2}k = 2.$

$\therefore k = 4$

We can find y_c by trying a solution $y_t = A(-c)^t$

I.e. $y_t = A(\dfrac{1}{2})^t$, where A is an arbitrary constant.

The general solution to the difference equation is therefore: $y_t = 4 + A(\dfrac{1}{2})^t = 4 + 2^{-t} A$

It remains to use the boundary condition $y_0 = 2$ to find *A*. Setting *t* =0 in the general solution we have

$$2 = 4 + A$$
$$\therefore A = 2$$
$$and$$
$$y_t = 4 - 2^{1-t}$$

Second-Order Difference Equations

A second-order difference equation is one that involves an expression $\Delta^2 y_t$, called the second-difference of y_t, but contains no differences of order higher than 2. The symbol Δ^2 is an instruction to take the second-difference as follows:

$$\Delta^2 y_t = \Delta(\Delta y_t) = \Delta(y_{t+1} - y_t)$$
$$= (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t)$$
$$= y_{t+2} - 2y_{t+1} + y_t$$

A simple variety of second-order difference equation takes the form:

(1) $y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$

This equation is linear, non-homogenous and with constant coefficients a_1, a_2 and constant term *c*.

As before, the solution has two components: a *particular* solution y_p, and a *complementary* function y_c,

The particular solution – defined as any solution of the complete equation – can be found simply by trying a solution of the form $y_t = k$. Substituting this value into (1) above, gives:

$$k + a_1 k + a_2 k = c$$
$$\therefore k = \frac{c}{1 + a_1 + a_2}$$

Thus, so long as $(1 + a_1 + a_2) \neq 0$, the particular solution is:

$$y_p (= k) = \frac{c}{1 + a_1 + a_2}, \text{ where } a_1 + a_2 \neq -1$$

Example: Find the particular solution of

$$y_{t+2} - 3y_{t+1} + 4y_t = 6.$$
$$Here\ a_1 = -3,\ a_2 = 4,\ c = 6$$
$$Since\ \ a_1 + a_2 \neq -1, y_p = \frac{6}{1 - 3 + 4} = 3$$

Case where $a_1 + a_2 = -1$:

Here the trial solution $y_t = k$ breaks down and it is necessary to try $y_t = kt$ instead.
Substituting this into (1) above and keeping in mind that we now have:

$$y_{t+1} = k(t+1)$$
$$y_{t+2} = k(t+2)$$
$$\therefore k(t+2) + a_1 k(t+1) + a_2 kt = c$$
$$and$$
$$k = \frac{c}{(1+a_1+a_2)t + a_1 + 2} = \frac{c}{a_1+2} \quad [\text{since } a_1 + a_2 = -1]$$

Thus we can write the particular solution as $y_p (= kt) = \dfrac{c}{a_1+2}t$, where $a_1 + a_2 = -1, a_1 \neq -2$.

Example: Find the particular solution of $y_{t+2} + y_{t+1} - 2y_t = 12.$

The Complementary Solution:

To find the complementary function, we must concentrate on the reduced equation:
(2) $y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$

The solution procedure involves solving the for the roots of the characteristic equation. In this case the characteristic equation is $b^2 + ba_1 + a_2 = 0$ and it possesses two characteristic roots:

$$b_1, b_2 = \frac{-a_1 \pm \sqrt{a_1^{\,2} - 4a_2}}{2}.$$

Three possible situations may be encountered with regard to the characteristic roots:

1. Distinct real roots, i.e. $b_1 \neq b_2$.

   If the characteristic equation has got two distinct real roots, the solution is:

   $y_c = A_1 b_1^{\,t} + A_2 b_2^{\,t}$, where A_1 and A_2 are constants

2. Repeated real roots, i.e. $b_1 = b_2$

   If the characteristic equation has co-incident roots, $b_1 = b_2 = b = -\dfrac{a_1}{2}$, then the solution is as follows:

   $$y_c = A_1 b^t + A_2 b^t = (A_1 + A_2) b^t = A_3 b^t$$

   BUT since the two components have collapsed into a single term, we are short of a constant. We therefore have to supply the missing component. We use the trick of multiplying b^t by the variable *t*.

   The complementary function for the repeated-root case is therefore:

$$y_c = A_3 b^t + A_4 t b^t$$

3. Complex roots.
   If $a_1^2 < 4a_2$, we have complex conjugate roots. We won't deal with this possibility in this course.

SYSTEMS OF DIFFERENTIAL & DIFFERENCE EQUATIONS

Refer to Chiang, Chapter 18

So far we have solved only stand-alone differential and difference equations. However, we may be confronted with simultaneous differential and difference equations.

When?

These arise from a set of interacting changes, e.g. in multi-sector markets when changes in one market affects conditions in another.

To deal with systems of equations, we have to transform higher-order equations into a more manageable form (i.e. all of the first-order).

Suppose we have:

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$$

Then we can let:

$$x_t = y_{t+1}$$
$$\Rightarrow x_{t+1} = y_{t+2}$$

This gives us:

$$x_{t+1} + a_1 x_t + a_2 y_t = c$$
$$x_t - y_{t+1} = 0$$

Similarly we can transform a n^{th} order differential equation into a system of n first-order equations. Given the differential equation

$y''(t) + a_1 y'(t) + a_2 y(t) = 0$ we introduce a new variable *x(t)* defined by

$$x(t) = y'(t) \qquad [\text{implying } x'(t) = y''(t)]$$

Then we can re-write the differential equation as the following system of two first-order equations:

$$x'(t) + a_1 x(t) + a_2 y(t) = 0$$
$$y'(t) - x(t) = 0$$

We therefore have to transform any higher order difference/ differential equation into a system of first-order difference or differential equations.

Solving Systems of Dynamic Equations:

The methods for solving simultaneous differential equations and simultaneous difference equations are quite similar. We'll only consider linear equations with constant coefficients.

1. Simultaneous Difference Equations

Suppose we have

$$x_{t+1} + 6x_t + 9y_t = 4$$
$$y_{t+1} - x_t = 0$$

Since **particular solutions** represent intertemporal equilibrium values, let us denote them by $\bar{x}$ and $\bar{y}$. As before we first try constant solutions, namely $x_{t+1} = x_t = \bar{x}$ and $y_{t+1} = y_t = \bar{y}$.

This works in the present case, because when we substitute these into the system of equations we get:

$$7\bar{x} + 9\bar{y} = 4$$
$$-\bar{x} + \bar{y} = 0$$
$$\Rightarrow \bar{x} = \bar{y} = \frac{1}{4}$$

(If these constant solutions don't work then you have to try solutions of the form $x_t = k_1 t,\ y_t = k_2 t$ etc.

For the complementary functions, we should – using our previous experience – adopt trial solutions of the form:

$x_t = mb^t$ and $y_t = nb^t$, where n and m are arbitrary constants and the base b represents the characteristic root.

It is automatically implied that $x_{t+1} = mb^{t+1}$ and $y_{t+1} = nb^{t+1}$.
To simplify matters we are using the same base b for both variables, although their coefficients are allowed to differ.

It is our aim to find the values of *b, m* and *n* that can make the trial solutions above satisfy the reduced homogenous version of $\begin{matrix} x_{t+1} + 6x_t + 9y_t = 4 \\ y_{t+1} - x_t = 0 \end{matrix}$.

Upon substituting the trial solutions into the reduced versions of $\begin{matrix} x_{t+1} + 6x_t + 9y_t = 4 \\ y_{t+1} - x_t = 0 \end{matrix}$ and cancelling the common factor $b^t \neq 0$, we obtain two equations:

$$(b+6)m+9n=0$$
$$-m+bn=0$$

This can be considered as a linear homogenous-equation system in two variables *m* and *n* (considering *b* as a parameter for the time being).

Rule out the trivial solution of $m = n = 0$, by requiring the coefficient matrix of the system to be singular. That is, we require the determinant of the matrix to be equal to zero:

$$\det\begin{pmatrix} b+6 & 9 \\ -1 & b \end{pmatrix} = b^2+6b+9=0$$
$$\therefore b(=b_1=b_2)=-3$$

The above equation is called the characteristic equation and its roots the characteristic roots, of the given simultaneous difference-equation system.

Given the value of *b*, we can get the values of *m* and *n* from $\begin{matrix}(b+6)m+9n=0 \\ -m+bn=0\end{matrix}$.

Since the system is homogenous, an infinite number of solutions for *(m, n)* will emerge, expressible in the form of the equation $m = kn$, where *k* is a constant.

In fact, for each root b_i, there will in general be a distinct equation $m_i = k_i n_i$,. Even with repeated roots, we should still use two such equations $m_1 = k_1 n_1$, and $m_2 = k_2 n_2$ in the complementary functions.

Moreover, with repeated roots, we recall that we can write the complementary functions as:

$$x_t = m_1(-3)^t + m_2 t(-3)^t$$
$$y_t = n_1(-3)^t + n_2 t(-3)^t$$

The factors of proportionality between m_i and n_i must of course satisfy the given equation system which mandates that $y_{t+1} = x_t$, i.e.

$$n_1(-3)^{t+1} + n_2(t+1)(-3)^{t+1} - m_1(-3)^t + m_2 t(-3)^t$$

Dividing through by $(-3)^t$:

$$-3n_1 - 3n_2(t+1) = m_1 + m_2 t$$
$$or$$
$$-3(n_1+n_2) - 3n_2 t = m_1 + m_2 t$$

Equating terms with *t* on two sides of the equals sign and similarly for the terms without *t,* we find that

$m_1 = -3(n_1 + n_2)$

and

$m_2 = -3n_2$

If we now write $n_1 = A_3$ and $n_2 = A_{4,}$ then it follows that:

$m_1 = -3(A_3 + A_4)$

$m_2 = -3A_4$

Thus the complementary functions can be written as:

$$x_c = -3(A_3 + A_4)(-3)^t - 3A_4 t(-3)^t$$

$$= -3A_3(-3)^t - 3A_4(t+1)(-3)^t$$

and

$$y_c = A_3(-3)^t + A_4 t(-3)^t$$

2. Simultaneous Differential Equations

The solution to a first-order linear differential equation is similar. The only major modification is to change the trial solutions to:

$$x(t) = me^{rt}$$

and

$$y(t) = ne^{rt}$$

This implies that $x'(t) = rme^{rt}$ and $y'(t) = rne^{rt}$.