Mathematics for Economists

Eigenvalues & Eigenvectors

Section 5: Dynamic Analysis

Eigenvalues and Eigenvectors

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This is an important topic in linear algebra. We will lay the foundation for the discussion of dynamic systems. We will also discuss some properties of symmetric matrices that are useful in statistics and economics.

1 Diagonalisable matrices

We can calculate powers of a square matrix, $\mathbf{A}$:

$$\mathbf{A}^2 = \mathbf{AA},\ \mathbf{A}^3 = \mathbf{A}^2\mathbf{A},\ \mathbf{A}^4 = \mathbf{A}^3\mathbf{A}$$

Matrix multiplication is associative, so $\mathbf{A}^{r+s} = \mathbf{A}^r\mathbf{A}^s$, so we could calculate $\mathbf{A}^{11}$ as $\mathbf{A}^6\mathbf{A}^5 = \left(\mathbf{A}^3\mathbf{A}^3\right)\left(\mathbf{A}^3\mathbf{A}^2\right)$.

Is there a general relation that holds between the entries of $\mathbf{A}^k$ and those of $\mathbf{A}$?

First, note how simple it is to calculate powers of a diagonal matrix:
If $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, then
$\mathbf{A}^2 = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & d^2 \end{bmatrix}$, $\mathbf{A}^3 = \begin{bmatrix} a^2 & 0 \\ 0 & d^2 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a^3 & 0 \\ 0 & d^3 \end{bmatrix}$, etc.

We use the following notation for diagonal matrices:

$\text{diag}\left(d_1, d_2, ..., d_n\right)$ is the diagonal matrix whose diagonal entries are $d_1, d_2, ..., d_n$.

Example 1 $\text{diag}(a, d) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, $\qquad \text{diag}(-7, 0, 6) = \begin{bmatrix} -7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

Powers of diagonal matrices obey the following rule:

$$\text{If } \mathbf{D} = \text{diag}\left(d_1, d_2, ..., d_n\right) \text{ then } \mathbf{D}^k = \text{diag}\left(d_1^k, d_2^k, ..., d_n^k\right) \tag{1}$$

Definition 1 *A square matrix* $\mathbf{A}$ *is* diagonalisable *if, for some invertible matrix* $\mathbf{S}$, $\mathbf{S}^{-1}\mathbf{AS}$ *is a diagonal matrix.*

We will use the phrase d-matrix to mean "diagonalisable matrix".

Every diagonal matrix is a d-matrix (let $\mathbf{S} = \mathbf{I}$), but there are many diagonalisable matrices that are not diagonal matrices.

In the case where $\mathbf{A}$ is a d-matrix, we can find a neat formula relating $\mathbf{A}^k$ to $\mathbf{A}$:

Remember $\mathbf{A}$ is diagonalisable if and only if $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ for some invertible matrix $\mathbf{S}$ and some diagonal matrix $\mathbf{D}$.

So

$$\mathbf{SDS}^{-1} = \mathbf{SS}^{-1}\mathbf{ASS}^{-1} = \mathbf{IAI} = \mathbf{A}.$$

It follows that

$$\begin{aligned}
\mathbf{A}^2 &= \left(\mathbf{SDS}^{-1}\right)\left(\mathbf{SDS}^{-1}\right) = \mathbf{S}\left(\mathbf{DID}\right)\mathbf{S}^{-1} = \mathbf{SD}^2\mathbf{S}^{-1}, \\
\mathbf{A}^3 &= \mathbf{A}^2\mathbf{A} = \left(\mathbf{SD}^2\mathbf{S}^{-1}\right)\left(\mathbf{SDS}^{-1}\right) = \mathbf{S}\left(\mathbf{D}^2\mathbf{ID}\right)\mathbf{S}^{-1} = \mathbf{SD}^3\mathbf{S}^{-1}
\end{aligned}$$

etc. Thus,

$$\text{If } \mathbf{A} = \mathbf{SDS}^{-1} \text{ then } \mathbf{A}^k = \mathbf{SD}^k\mathbf{S}^{-1} \text{ for } k = 1, 2, ... \tag{2}$$

This applies only to d-matrices. Two questions now arise:

1. How can we tell if a given matrix is diagonalisable?
2. If it is, how do we find $\mathbf{S}$ and $\mathbf{D}$?

To answer these questions we introduce the concepts of eigenvalues and eigenvectors.

1.1 The key definitions

Definition 2 *Let* $\mathbf{A}$ *be a square matrix,* λ *a scalar. We say that* λ *is an* eigenvalue *of* $\mathbf{A}$ *if there exists a vector* $\mathbf{x}$ *such that* $\mathbf{x} \neq 0$ *and*

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Such an $\mathbf{x}$ *is called an* eigenvector *of* $\mathbf{A}$ *corresponding to the eigenvalue* λ*.*

By definition, the scalar λ is an eigenvalue of the matrix $\mathbf{A}$ if and only if $(\lambda\mathbf{I} - \mathbf{A})\,\mathbf{x} = \mathbf{0}$ for some non-zero vector $\mathbf{x}$; in other words, λ is an eigenvalue of $\mathbf{A}$ if and only if $(\lambda\mathbf{I} - \mathbf{A})$ is a singular matrix. Thus the eigenvalues of $\mathbf{A}$ can in principle be found by solving the equation

$$\det\left(\lambda\mathbf{I} - \mathbf{A}\right) = 0. \tag{3}$$

If $\mathbf{A}$ is a 2×2 matrix, then (3) is a quadratic equation in λ.

Example 2 *Find the eigenvalues and eigenvectors of the matrix*

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$$

We find the eigenvalues using (3):

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda - 3 & 1 \\ -4 & \lambda + 2 \end{vmatrix} = (\lambda - 3)(\lambda + 2) + 4.$$

Hence the eigenvalues of $\mathbf{A}$ *are the roots of the quadratic equation*

$$\lambda^2 - \lambda - 2 = 0$$

The eigenvalues are therefore 2 *and* -1.
We now find the eigenvectors of $\mathbf{A}$ *corresponding to the eigenvalue* 2.
$\mathbf{Ax} = 2\mathbf{x}$ *if and only if*

$$\begin{aligned} 3x_1 - x_2 &= 2x_1 \\ 4x_1 - 2x_2 &= 2x_2 \end{aligned}$$

Each of these equations simplifies to $x_1 = x_2$. *(It should not come as a surprise to you that the two equations collapse to one since we have just shown that* $(2\mathbf{I} - \mathbf{A})$ *is singular.)*

Thus the eigenvectors corresponding to the eigenvalue 2 *are the non-zero multiples of* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Similarly, $\mathbf{Ax} = -\mathbf{x}$ *if and only if* $x_2 = 4x_1$; *thus the eigenvectors of* $\mathbf{A}$ *corresponding to the eigenvalue* -1 *are the non-zero multiples of* $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

1.1.1 Terminology

There are many synonyms for eigenvalue, including *proper value*, *characteristic root* and *latent root*. Similarly, there are many synonyms for eigenvector.

The 'root' arises because the eigenvalues are the solutions ('roots') of a polynomial equation.

1.2 Three propositions

The following connect eigenvalues and eigenvectors with diagonalisability.

Proposition 1 *An* $n \times n$ *matrix* $\mathbf{A}$ *is diagonalisable if and only if it has* n *linearly independent eigenvectors.*

Proof. *Suppose the $n \times n$ matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{x}^1, ..., \mathbf{x}^n$ corresponding to eigenvalues $\lambda_1, ..., \lambda_n$ respectively. Then*

$$\mathbf{A}\mathbf{x}^j = \lambda_j \mathbf{x}^j \text{ for } j = 1, ..., n \tag{4}$$

Define the two $n \times n$ matrices

$$\mathbf{S} = \begin{pmatrix} \mathbf{x}^1 & \dots & \mathbf{x}^n \end{pmatrix}, \qquad \mathbf{D} = diag\,(\lambda_1, ..., \lambda_n) \tag{5}$$

Since $\mathbf{x}^1, ..., \mathbf{x}^n$ are linearly independent, $\mathbf{S}$ is invertible. By (4),

$$\mathbf{AS} = \mathbf{SD} \tag{6}$$

Premultiplying (6) by $\mathbf{S}^{-1}$ we have $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D}$ so $\mathbf{A}$ is indeed a d-matrix.

Conversely, suppose that $\mathbf{A}$ is a d-matrix. Then we may choose an invertible matrix $\mathbf{S}$ and a diagonal matrix $\mathbf{D}$ satisfying (6). Given the properties of $\mathbf{S}$ and $\mathbf{D}$, we may define vectors $\mathbf{x}^1, ..., \mathbf{x}^n$ and scalars $\lambda_1, ..., \lambda_n$ satisfying (5); then (6) may be written in the form (4). Since $\mathbf{S}$ is invertible the vectors $\mathbf{x}^1, ..., \mathbf{x}^n$ are linearly independent: in particular none of them is the zero-vector. Hence by (4), $\mathbf{x}^1, ..., \mathbf{x}^n$ are n linearly independent eigenvectors of $\mathbf{A}$. ■

Proposition 2 *If $\mathbf{x}^1, ..., \mathbf{x}^k$ are eigenvectors corresponding to k different eigenvalues of the $n \times n$ matrix $\mathbf{A}$, then $\mathbf{x}^1, ..., \mathbf{x}^k$ are linearly independent.*

Proposition 3 *An $n \times n$ matrix $\mathbf{A}$ is diagonalisable if it has n different eigenvalues.*

1.2.1 Applying the propositions

Given the eigenvalues of a square matrix $\mathbf{A}$,

- Proposition 3 provides a sufficient condition for $\mathbf{A}$ to be a d-matrix.

If this condition is met

- Proposition 2 and the proof of Proposition 1 show how to find $\mathbf{S}$ and $\mathbf{D}$.

We can then use

- (1) and (2) to find $\mathbf{A}^k$ for any positive integer k.

Example 3 *As in Example 2 let*

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$$

We show that $\mathbf{A}$ *is a d-matrix, and find* $\mathbf{A}^k$ *for any positive integer* k*.*

From Example 2, we know that the eigenvalues are 2 *and* -1*. Thus* $\mathbf{A}$ *is a* 2×2 *matrix with two different eigenvalues; by Proposition 3* $\mathbf{A}$ *is a d-matrix.*

Now let $\mathbf{x}$ *and* $\mathbf{y}$ *be eigenvectors of* $\mathbf{A}$ *corresponding to the eigenvalues* 2 *and* -1 *respectively. By Proposition 2,* $\mathbf{x}$ *and* $\mathbf{y}$ *are linearly independent. Hence, by the proof of Proposition 1 we may write* $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\mathbf{D}$ *where* $\mathbf{D}=diag(2,-1)$ *and* $\mathbf{S}=\begin{pmatrix}\mathbf{x} & \mathbf{y}\end{pmatrix}$*. Now* $\mathbf{x}$ *and* $\mathbf{y}$ *can be any eigenvectors corresponding to the eigenvalues* 2 *and* -1 *respectively: so by the results of Example 2 we may let* $\mathbf{x}=\begin{bmatrix}1\\1\end{bmatrix}$ *and* $\mathbf{y}=\begin{bmatrix}1\\4\end{bmatrix}$*.*

Summarising,

$$\mathbf{S}=\begin{bmatrix}1 & 1\\1 & 4\end{bmatrix},\qquad \mathbf{D}=\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}$$

To find $\mathbf{A}^k$ *using (1) and (2), we must first calculate* $\mathbf{S}^{-1}$*.*

$$\mathbf{S}^{-1}=\frac{1}{3}\begin{bmatrix}4 & -1\\-1 & 1\end{bmatrix}$$

Hence, by (1) and (2),

$$\mathbf{A}^k=\begin{bmatrix}1 & 1\\1 & 4\end{bmatrix}\begin{bmatrix}2^k/3 & 0\\0 & (-1)^k/3\end{bmatrix}\begin{bmatrix}4 & -1\\-1 & 1\end{bmatrix}\text{ for } k=1,2,...$$

You could multiply this expression out if you want.

1.2.2 Some remarks

Remark 1 *When we write a d-matrix in the form* $\mathbf{SDS}^{-1}$*, we have some choice about how we write* $\mathbf{S}$ *and* $\mathbf{D}$*. For example, in Example 3 we could have chosen the second column of* $\mathbf{S}$ *to be* $\begin{bmatrix}\frac{1}{2}\\2\end{bmatrix}$*. The one rule that must be strictly followed is: the first column in* $\mathbf{S}$ *must be the eigenvector* corresponding to the eigenvalue *that is the first diagonal entry of* $\mathbf{D}$*, and similarly for the other columns.*

Remark 2 *Proposition 1 gives a necessary and sufficient condition for a matrix to be diagonalisable. Proposition 3 gives a sufficient but not necessary condition for a matrix to be diagonalisable. In other words, any* $n\times n$ *matrix with* n *different eigenvalues is a d-matrix, but there are* $n\times n$ *d-matrices which do not have* n *different eigenvalues.*

1.3 Diagonalisable matrices with non-distinct eigenvalues

The n eigenvalues of an $n\times n$ matrix are not necessarily all different. There are two ways of describing the eigenvalues when they are not all different. Consider a 3×3 matrix $\mathbf{A}$ where

$$\det(\lambda\mathbf{I}-\mathbf{A})=(\lambda-6)^2(\lambda-3)=0$$

we say either

- the eigenvalues of $\mathbf{A}$ are $6, 6, 3$; or
- $\mathbf{A}$ has eigenvalues 6 (with *multiplicity* 2) and 3 (with multiplicity 1).

Proposition 4 *If the $n \times n$ matrix $\mathbf{A}$ does not have n distinct eigenvalues but can be written in the form $\mathbf{SDS}^{-1}$, where $\mathbf{D}$ is a diagonal matrix,* the number of times each eigenvalue occurs on the diagonal of $\mathbf{D}$ is equal to its multiplicity.

Example 4 *We show that the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 4 \end{bmatrix}$$

is diagonalisable.

Expanding $\det(\lambda\mathbf{I} - \mathbf{A})$ *by its second row, we see that*

$$\det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - 2)\left(\lambda^2 - 5\lambda + 6\right) = (\lambda - 2)^2(\lambda - 3)$$

Setting $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$, *we see that the eigenvalues are* $2, 2, 3$. *If $\mathbf{A}$ is to be a d-matrix, then the eigenvalue 2 must occur twice on the diagonal of $\mathbf{D}$, and the corresponding columns of $\mathbf{S}$ must be two linearly independent eigenvectors corresponding to this eigenvalue. Now the eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue 2 are those non-zero vectors $\mathbf{x}$ such that $x_1 = 2x_3$. Two linearly independent vectors of this type are:*

$$\mathbf{s}^1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{s}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence $\mathbf{A}$ is a d-matrix, and $\mathbf{D} = \text{diag}(2, 2, 3)$ *and the first two columns of $\mathbf{S}$ are $\mathbf{s}^1$ and $\mathbf{s}^2$. The third column of $\mathbf{S}$ is an eigenvector corresponding to the eigenvalue 3, such as* $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Summarising, $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D}$ *where*

$$\mathbf{S} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

2 Complex linear algebra

Example 5 *Find the eigenvalues and eigenvectors of the matrix*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

We find the eigenvalues by solving the characteristic equation

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}) &= \lambda^2 + 1 = 0 \\ &\Rightarrow \lambda = \pm i \end{aligned}$$

We now find the eigenvectors of $\mathbf{A}$ *corresponding to the eigenvalue* i*.*
$\mathbf{Ax} = i\mathbf{x}$ *if and only if*

$$x_2 = ix_1$$

Thus the eigenvectors corresponding to the eigenvalue i *are the non-zero multiples of* $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Similarly, $\mathbf{Ax} = -i\mathbf{x}$ *if and only if* $x_1 = ix_2$*; thus the eigenvectors of* $\mathbf{A}$ *corresponding to the eigenvalue* $-i$ *are the non-zero multiples of* $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

3 Trace and determinant

It can be shown that

$$\begin{aligned} \text{tr } \mathbf{A} &= \lambda_1 + \lambda_2 + ... + \lambda_n \\ \det \mathbf{A} &= \lambda_1 \lambda_2 ... \lambda_n \end{aligned} \tag{7}$$

i.e. *the trace is the sum of the eigenvalues and the determinant is the product of the eigenvalues.*

These statements are true whether or not the matrix is diagonalisable.

Example 6 *Consider the matrix from Example 4:*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 4 \end{bmatrix}$$

We showed that the eigenvalues of $\mathbf{A}$ *are* $2, 2, 3$*.*
The trace is the sum of the diagonal entries

$$\text{tr } \mathbf{A} = \mathbf{1} + \mathbf{2} + \mathbf{4} = \mathbf{7}$$

which is equal to the sum of the eigenvalues

$$\text{tr }\mathbf{A} = \mathbf{2} + \mathbf{2} + \mathbf{3} = \mathbf{7}.$$

The determinant is (expanding by the second row)

$$\det \mathbf{A} = \mathbf{2}(\mathbf{4} + \mathbf{2}) = \mathbf{12}$$

which is equal to the product of the eigenvalues

$$\det \mathbf{A} = (\mathbf{2})(\mathbf{2})(\mathbf{3}) = \mathbf{12}.$$

4 Non-diagonalisable matrices

Not all square matrices are diagonalisable.

Example 7 *Let*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The characteristic polynomial is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^2$$

Setting this equal to zero, we find that the eigenvalues are $0, 0$. *The associated eigenvectors are the non-zero multiples of* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

But then $\mathbf{A}$ *is a* 2×2 *matrix which does not have* 2 *linearly independent eigenvectors; so by Proposition 1 of Section 1.2,* $\mathbf{A}$ *is not diagonalisable.*

However, almost all square matrices are diagonalisable. In most practical contexts the assumption that the relevant square matrices are diagonalisable does not involve much loss of generality.

5 Eigenvalues of symmetric matrices

Definition 3 *A* symmetric matrix *is a square matrix whose transpose is itself. So the* $n \times n$ *matrix* $\mathbf{A}$ *is symmetric if and only if*

$$\mathbf{A} = \mathbf{A}'$$

Theorem 1 *If* $\mathbf{A}$ *is a symmetric matrix,* $\mathbf{A}$ *is diagonalisable - there exist a diagonal matrix* $\mathbf{D}$ *and an invertible matrix* $\mathbf{S}$ *such that* $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$;

5.1 Definiteness

We now describe positive definite, positive semidefinite, negative definite, negative semidefinite matrices in terms of their eigenvalues.

Theorem 2 *A symmetric matrix* $\mathbf{A}$ *is*

- *positive definite if and only if all its eigenvalues are positive;*
- *positive semidefinite if and only if all its eigenvalues are non-negative;*
- *negative definite if and only if all its eigenvalues are negative;*
- *negative semidefinite if and only if all its eigenvalues are non-positive.*

Example 8 *Let* $\mathbf{A}$ *be the symmetric matrix*

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

The characteristic polynomial is:

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^2 - 8\lambda + 15$$

Setting this equal to zero, we see that the eigenvalues are 5 *and* 3
Since these are positive numbers, $\mathbf{A}$ *is positive definite.*

References

[1] Pemberton, M. and Rau, N.R. 2001. *Mathematics for Economists: An introductory textbook*, Manchester: Manchester University Press.