Mathematics for Economists
Integration
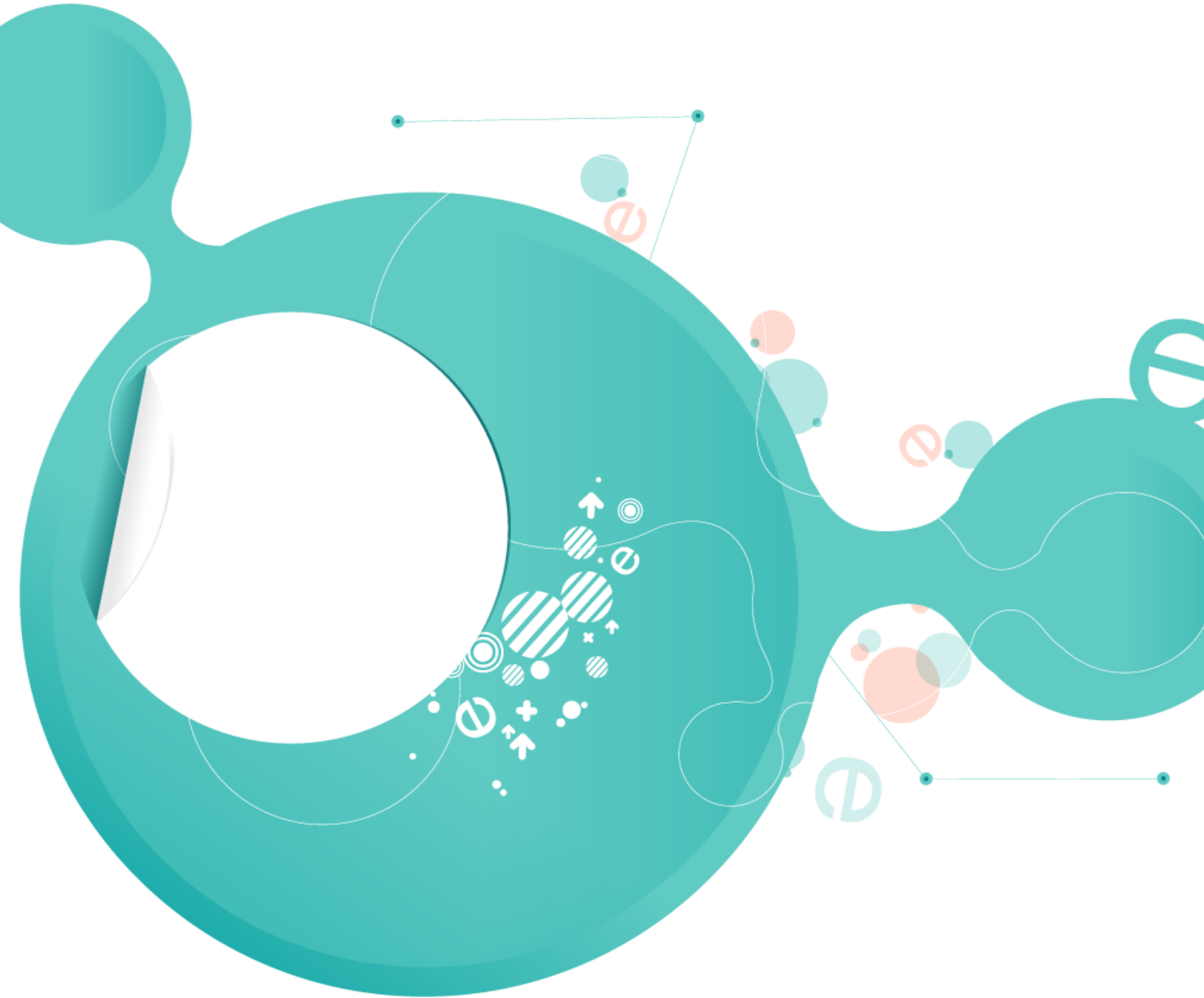
Reading: Chiang Chapter 14

Note: These notes do not fully cover the material in Chiang, but are meant to supplement your reading in Chiang.

Thus far the optimisation you have covered has been static in nature, that is optimising the value of a function without any reference to time. In static optimisation and comparative static analysis we make the assumption that the process of economic adjustment leads to an equilibrium, and we then examine the effect of changes of the exogenous variables on the equilibrium values of the endogenous variables. With dynamic analysis, time is explicitly considered in the analysis. While we are not covering dynamic analysis at this point, certain mathematic tools are required for dynamic analysis, such as integration and differential equations. Without these tools, it becomes impossible to consider problems which are not static in nature. We will be covering both of these topics in a mainly mathematical way, leaving economic problems for a later date.

Integration is the reverse process of differentiation. If a function $F(x)$ has first derivative $f(x)$ then the integral of $f(x)$ will yield $F(x)$. The notation to denote integration is as follows:

$\int f(x)dx$, where the integral sign is an elongated S. $f(x)$ is referred to as the integrand, and the dx sign reminds us that we are integrating with respect to the variable x. We go through the following explanation to determine where this notation comes from.

Suppose we are given an arbitrary function $f(x)$ and asked to find the area of the curve between 2 points, for example the area under the curve $f(x)$ between 5 and 10.
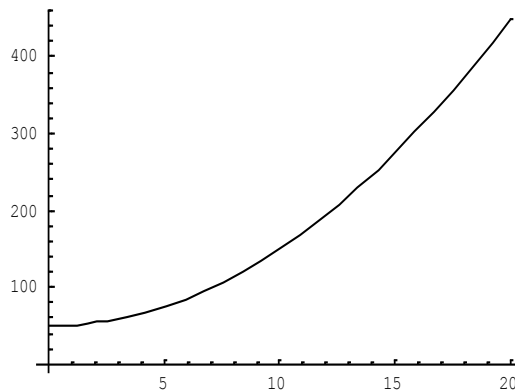

Figure 1.1

With a linear function, this equates to finding the area of a triangle and a rectangle as follows:

Figure 1.2

However with a non-linear function the problem becomes slightly more complex. What we can do, however, is attempt to find the area under the curve using a number of approximating rectangles as follows:
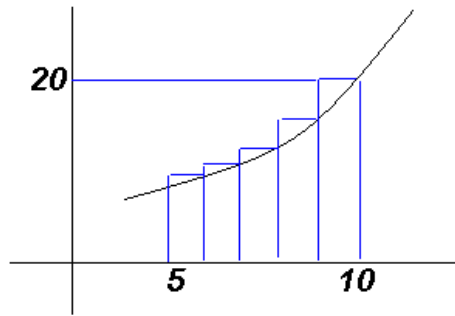

Figure 1.3

We let each of the rectangles have equal width and we call this width Δx. Each rectangle has a height equal to the function value, for example the height of the last rectangle where $x=10$ is equal to $f(10)=20$. Thus the area of the last rectangle is equal to $20(\Delta x)$, as area of a rectangle equals length times breadth, and here breadth is Δx and length is $f(10)=20$. The area of any of the rectangles is equal to length times breadth, which equals $f(x)$ times Δx, as all rectangles have equal breadth equal to Δx. To find the area under the curve we add up the area of each of the rectangles. This gives us the expression:

$$A = \sum_{i=1}^{n} f(x_i)\Delta x$$

Where
n = number of rectangles
x_i = the value of x at each point
$\sum$ = the sum of all the areas, starting from the first one ($i = 1$) and ending at the nth one ($i = 1$).

Obviously this sum will not be a very accurate representation of the area. But perhaps if we make our Δx smaller, then this expression will become a more accurate representation of the area under the curve, as there will be less overshooting by each rectangle. If initially we had ten rectangles, the area given by the sum of these rectangles' areas would obviously be more of an over-estimate (or maybe underestimate) than if we doubled the number of rectangles, and then summed their area. The more rectangles we use in this approximating process the better our estimate for the area under the curve.

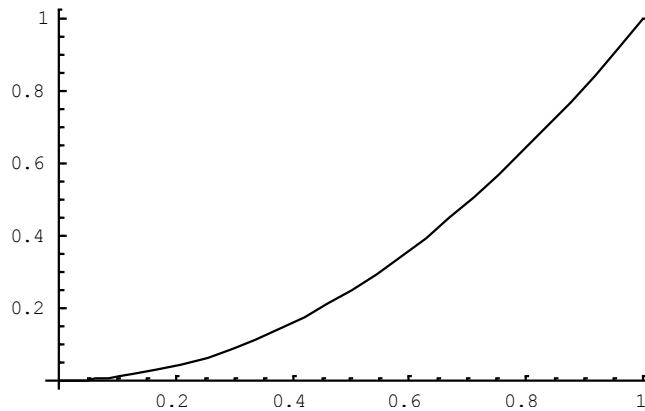For example, imagine we wish to find the area under the curve $f(x) = x^2$ between 0 and 1.

Figure 1.4

2

We could take four rectangles, each with a breadth Δx equal to 0.25, and take the heights from the right hand side of each rectangle. Hence the height for each rectangle will be:

$(0.25)^2$

$(0.5)^2$

$(0.75)^2$

1^2

Therefore the entire area equals to

$0.25(0.25)^2 + 0.25(0.5)^2 + 0.25(0.75)^2 + 0.25(1)^2 = 15/32 = 0.46875$

If we double the amount of rectangles from four to eight, we will use a Δx of 0.125, and the following right hand heights (remember the height of the rectangle is given by the function value $f(x)$).

$(0.125)^2$ $(0.25)^2$

$(0.375)^2$ $(0.5)^2$

$(0.625)^2$ $(0.75)^2$

$(0.875)^2$ 1^2

The corresponding total area is given by the sum of each of the areas which is Δx multiplied by each function value: The final value we get is 0.3984375

As can be seen in figure 1.3, using right end points for the rectangles for an increasing function will give an over-estimate, while using right end points for a decreasing function will yield an over-estimate. Thus doubling the number of rectangles while trying to estimate the area under the graph $f(x)=x^2$ will begin to bring our estimate down to its true value. It appears that as the number of rectangles increases, our estimations become better and better approximations of the area. If we let the number of rectangles tend to infinity, we will obtain a perfectly accurate estimate for the area under our graph.

Our expression for the area under the curve now becomes:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

This gives us the expression for the definite integral, which gives us a way of finding the area under the continuous function $f(x)$ between $x=a$ and $x=b$:

$$\int_{a}^{b} f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

An explanation of the terminology:

$\int$ The integration sign is an elongated S, and was so chosen because an integral is a limit of sums.

a, b are the limits of integration,

a is the lower limit of integration

b is the upper limit of integration.

$f(x)$ is known as the integrand.

dx has no meaning by itself, but merely reminds us that we are integrating with respect to the variable x.

Fortunately when we want to find the area under a curve, we do not have to go into the long process of finding an expression for the sum of the area of n rectangles: a number of theorems make the process easier.

3

Before we set out the properties of the definite integral, some rules of integration are as follows: (see page 439 and onwards in Chiang for examples).

1. **The Power Rule**

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c \quad (n \neq -1)$$

2. **The Exponential Rule**

$$\int e^x dx = e^x + c$$

3. **The Logarithmic Rule**

$$\int \frac{1}{x} dx = \ln x + c \quad (x > 0)$$

Properties of the definite integral:

$$1. \int_a^b c\,dx = c(b-a)$$

$$2. \int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$3. \int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$4. \int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$5. \int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

Property 1 states that the integral of a constant function $y=c$ is the constant times the length of the interval, as seen in figure 1.5
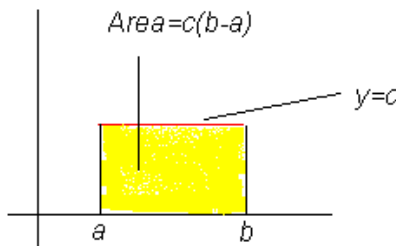

Figure 1.5

Property 2 says that the integral of a sum is the sum of the integrals. The area under $f+g$ is the area under f plus the area under g. This property follows from the property of limits and sums.

4

Property 3 tells us that a constant (but only a constant) can be taken in front of an integral sign. This also follows from the properties of limits and sums.

Property 4 follows from property 2 and 3, using *c=-1*.

Property 5 tells us we can find the area under the graph between *a* and *c*, by splitting it up into two areas, between *a* and *b*, and between *b* and *c*.

We now find our rule for evaluating the definite integral:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where the derivative of *F(x)* is *f(x)*, i.e. F is any anti-derivative of *f*.

For example, if we differentiate $F(x) = \dfrac{x^3}{3}$ we obtain $f(x) = x^2$, so F(x) is an anti-derivative of *f(x)*.

Thus $\int_0^1 x^2 dx = F(1) - F(0) = \dfrac{x^3}{3}\Big|_0^1 = \dfrac{1}{3} - 0 = \dfrac{1}{3}.$

Therefore the area under the curve *f(x) = x²* between 0 and 1, is equal to a third, or 0.33 recurring. Incidentally this answers our previous question which we attempted using the sum of the areas of n rectangles.

The fundamental theorem of calculus motivates this use of the evaluation theorem. In short, it states that differentiation and integration are opposite processes. Thus, if we start with a function *F(x)*, and differentiate it to obtain *f(x)*, [i.e. $F'(x) = f(x)$], if we then integrate the function *f(x)*, the result will be the initial function *F(x)*. Similarly if we integrate *f(x)* to obtain *F(x)*, [i.e. $\int f(x)dx = F(x) + C$ where C is an arbitrary constant[1]]. Thus to find the integral of a function *f(x)*, we must find the function which when differentiated yields *f(x)*. This theorem is very useful to us, as otherwise whenever we wish to find the value of the area that lies underneath a curve, we have to go through the entire process of finding the limit of the sum of the areas of n approximating rectangles, which is a time consuming process! Prior to the discovery of the fundamental theorem, finding areas, volumes and other similar types of problems were nigh on impossible.

For completeness, the fundamental theorem is presented below:

The fundamental theorem of calculus:

Suppose *f(x)* is a continuous function on the closed interval [*a,b*]

1. If $g(x) = \int_a^x f(t)dt$ then $g'(x) = f(x)$ i.e. $g'(x) = \dfrac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$

2. $\int_a^b f(x)dx = F(b) - F(a)$

where the derivative of *F(x)* is *f(x)*, i.e. F is any anti-derivative of *f*. What it says, roughly speaking, is that if you integrate a function and then differentiate the result, you retrieve the original function.

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[1] More about the arbitrary constant a little later

5

We now need to discuss two different types of integrals – definite and indefinite. A definite integral involves finding the integral of a function between two number limits i.e. $\int_a^b f(x)dx$. The answer to a definite integral is a number, as we know according to the evaluation rule the answer to this is just the anti-derivative $F(x)$ evaluated between a and b, i.e. $F(b)$-$F(a)$. An indefinite integral yields a function of x as its answer (if we are integrating with respect to x). An indefinite integral is an integral of the form $\int f(x)dx$ (i.e without upper and lower limits) and the solution is

$$\int f(x)dx = F(x) + C$$

where C is an arbitrary constant which can take on any value.

The reason we include the arbitrary constant is illustrated in the following example.

Given the problem:

$\int 3x^2 dx$ a potential solution is x^3 as this is an antiderivative of the cubic function (If we differentiate x^3 we obtain $3x^2$. However $x^3 + 4$ is also a solution to this problem, as is $x^3 - 100$. This is because when differentiating these expressions, the constant differentiated moves to zero. So it would appear that the most general form to give the answer to this problem would be as follows:

$$\int 3x^2 dx = x^3 + C,$$

where C is an arbitrary constant.

Just a small note on arbitrary constants – when we add two together, we obtain a third one which has aggregated the first two, when multiplying, dividing adding or subtracting a number by/to/from an arbitrary constant, the result is just the arbitrary constant. However the arbitrary constant when multiplied by a function of x, will stay as just that:

$$\int f(x) + g(x)dx = F(x) + C_1 + G(x) + C_2 = F(x) + G(x) + C$$

where C_1 and C_2 are two arbitrary constants, and $F(x)$ and $G(x)$ are two anti-derivates of $f(x)$ and $g(x)$ respectively.

$$3\int f(x)dx = 3[F(x) + C] = 3F(x) + 3C = 3F(x) + C$$

However:

$$x\int f(x)dx = x[F(x) + C] = xF(x) + Cx$$

We now turn to some rules of integration (definite and indefinite) and then some examples.

1. $\int cf(x)dx = c\int f(x)dx$

2. $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

3. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$ (n cannot equal -1)

4. $\int \dfrac{1}{x}dx = \ln|x| + C$

5. $\int e^x dx = e^x + C$

6. $\int a^x dx = \dfrac{a^x}{\ln a} + C$

7. $\int \sin x dx = -\cos x + C$

8. $\int \cos x dx = \sin x + C$

Remember – to check the answer to any integration sum just differentiate it and you should arrive back at the original function.

Some examples:

1. $\int 3dx = 3x + C$

2. $\int dx = x + C$

3. $\int \sqrt{x^3}\,dx = \int x^{\frac{3}{2}}dx = \dfrac{2}{5}x^{\frac{5}{2}} + C$

4. $\displaystyle\int_1^2 \dfrac{1}{x^4}dx = \int_1^2 x^{-4}dx = \dfrac{x^{-3}}{-3}\Big|_1^2 = -\dfrac{1}{24} - (-\dfrac{1}{3}) = \dfrac{7}{24}$

5. $\int [10x^4 + \sin x]dx = \dfrac{10x^5}{5} - \cos x + C = 2x^5 - \cos x + C$

6. $\displaystyle\int_0^3 (x^3 - 6x)dx = \dfrac{x^4}{4} - 6\dfrac{x^2}{2}\Big|_0^3 = \dfrac{x^4}{4} - 3x^2\Big|_0^3 = \dfrac{81}{4} - 3(9) - 0 + 0 = -6.75$

7.

$$\int_1^9 \dfrac{2t^2 + t^2\sqrt{t} - 1}{t}dt = \int_1^9 (2t + t\sqrt{t} - t^{-1})dt = \int_1^9 (2t + t^{\frac{3}{2}} - t^{-1})dt = t^2 + \dfrac{2}{5}t^{\frac{5}{2}} - \ln t\Big|_1^9$$

$$= 81 + \dfrac{2}{5}(243) - \ln 9 - 1 - \dfrac{2}{5} + \ln 1$$

$$= 174.603$$

8. $\displaystyle\int_1^4 (t^2 - t - 6)dt = \dfrac{t^3}{3} - \dfrac{t^2}{2} - 6t\Big|_1^4 = -\dfrac{9}{2}$

9.

$$\int \dfrac{2x-3}{2x+5} = \int \dfrac{2x+5-5-3}{2x+5} = \int \dfrac{2x+5}{2x+5} - \dfrac{8}{2x+5} \qquad \textbf{(Note the trick here)}$$

$$= \int 1 - \dfrac{8}{2x+5}$$

$$= x - 8\left(\ln(2x+5)\dfrac{1}{2} + c \right)$$

$$= x - 4\ln(2x+5) + c$$

A few more useful properties:

$$1. \int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$2. \int_a^a f(x)dx = 0$$

We are going to be looking at two very useful techniques used in integration: use of substitution, and integration by parts. There are a whole host of other techniques which can be useful, however it is these two which are most useful to us in economics.

Integration using Substitution

We use substitution, when the integrand contains a function and its own derivative. i.e:

$$\int f'(x)f(x)dx$$

For example:

$$\int (x^3 + 5x^2 - 10)(3x^2 + 10x)dx$$

If this is the case, we can make use of the following substitution: We let u equal to the function whose derivative we can spot (or create, using a constant: more about this later).

Let u = $x^3 + 5x^2 - 10$
Then we know that
$$du = (3x^2 + 10x)dx$$
When we then substitute the values of u and du into the integral, we obtain the following integral:
$$\int udu$$
which has the answer
$$\int udu = \frac{u^2}{2} + C$$
but: $u = x^3 + 5x^2 - 10$

Therefore our final answer is

$$\int (x^3 + 5x^2 - 10)(3x^2 + 10x)dx = \frac{(x^3 + 5x^2 - 10)^2}{2} + C$$

Thus the general rule solution for the problem is as follows:

$$\int f'(x)f(x)dx = \frac{[f(x)]^2}{2} + C \text{ or more simply } \int (f'.f)dx = \frac{f^2}{2} + C$$

However the substitution rule can be used for more complicated examples, when our function *f(x)* whose derivative we can spot occurs inside another function:

For example:

$$\int f'g(f)dx \quad \text{or} \quad \int (6x+18)\sqrt{(3x^2+18x-23)}dx$$

In these cases the procedure does not change at all – we still make the substitution as follows:
Let $u = f(x)$
Therefore $du=f'(x)dx$
And proceed as usual:

Some examples:

1. $\int (6x+18)\sqrt{(3x^2+18x-23)}dx$

Let $u = (3x^2+18x-23)$

Therefore $du = (6x+18)dx$

Therefore our transformed integral is as follows:

$$\int \sqrt{u}\,du = \int u^{\frac{1}{2}}du = \frac{2u^{\frac{3}{2}}}{3} + C = \frac{2(3x^2+18x-23)^{\frac{3}{2}}}{3} + C$$

2. Sometimes we can find a function whose derivative we can create as follows: However only if we introduce a constant function.

For example:

$$\int e^{(4x+5)}dx$$

If we let $u = (4x+5)$, we know $du = 4dx$. However while we have the dx, we do not have a 4. This is easily solved however through the following manipulation:

$$\frac{1}{4}\int 4e^{(4x+5)}dx$$

This integral now contains a function and its derivative, thus substitution can be used:
Therefore let $u = (4x+5)$, and $du = 4dx$. The integral becomes:

$$\frac{1}{4}\int e^u du = \frac{1}{4}e^u + C = \frac{1}{4}e^{(4x+5)} + C$$

Remember, the substitution of u into this function is a device that we employ. Therefore our final answer ought not to contain u, as the original problem does not contain it. Always remember to substitute back for u.

Note: Unlike differentiation, there exists no general formula giving the integral of a product of two function i.t.o. the separate integrals of those functions. There is also no general formula giving the integral of a quotient of 2 functions in terms of their separate integrals. As a result, integration is trickier than differentiation, on the whole.

Integration by Parts

We use this technique when we have to integrate a product:

Eg. $\int f(x)g'(x)dx$

When we are given this type of example, we make use of the following formula:

$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx}$$

For example:
$\int xe^x dx$

We pick *f* as the function which is easy to differentiate, and g' as the function which is easy to integrate. Often picking a squared or cubic term for your f is a good idea, as f' will have a power that is then one lower, and hence simpler. It is a very good idea to make yourself a mini table with f', g', f and g, to keep things straight. Also, note that when finding g, we do not bother with the arbitrary constant.

$\int xe^x dx$

Therefore:

$f = x$ $\qquad\qquad$ $g' = e^x$

$f' = 1$ $\qquad\qquad$ $g = e^x$

$\int xe^x dx = xe^x - \int (1)e^x dx$

$\qquad\qquad = xe^x - e^x + C$

Thus we have managed to use the formula to integrate our original question.

[Can also use the alternative notation used in Chiang: where $\int v\,du = uv - \int u\,dv$]

Another example:

$$\int x \sin x \, dx$$

Therefore:

$$f = x \qquad\qquad g' = \sin x$$
$$f' = 1 \qquad\qquad g = -\cos x$$

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx$$
$$= -x \cos x + \sin x + C$$

An example with a trick:

$$\int \ln x \, dx$$

Obviously we do not know the integral of *lnx*, that is why we are using this method. So we make *lnx* equal to *f*, and we can then find its derivative. But then what will our g' be? Simple, make it 1.

$$f = \ln x \qquad\qquad g' = 1$$
$$f' = \frac{1}{x} \qquad\qquad g = x$$

$$\int \ln x \, dx = x \ln x - \int x \left(\frac{1}{x} \right) dx$$
$$= x \ln x - x + C$$

This is a handy trick which can also be used to find the integrals of some of the trigonometric functions.

Some more examples:

$$\int x e^{2x} \, dx$$

Therefore:

$$f = x \qquad\qquad g' = e^{2x}$$
$$f' = 1 \qquad\qquad g = \frac{e^{2x}}{2}$$

$$\int xe^{2x}dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}dx$$

$$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}x + C$$

$$= \frac{e^{2x}}{2}\left(x - \frac{1}{2}\right) + C$$

Another example:

$$\int x\ln x\, dx$$

$$f = \ln x \qquad\qquad\qquad g' = x$$

$$f' = \frac{1}{x} \qquad\qquad\qquad g = \frac{x^2}{2}$$

$$\int x\ln x\, dx = \frac{x^2}{2}\ln x - \int\left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right)dx$$

$$= \frac{x^2}{2}\ln x - \int \frac{x}{2}dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

Another example:

$$\int (\ln x)^2 dx$$

$$f = (\ln x)^2 \qquad\qquad\qquad g' = 1$$

$$f' = 2(\ln x)\frac{1}{x} \qquad\qquad\qquad g = x$$

12

$$\int (\ln x)^2 \, dx = x(\ln x)^2 - \int (x)\left(2\ln x\left(\frac{1}{x}\right)\right)dx$$

$$= x(\ln x)^2 - \int 2\ln x \, dx$$

$$= x(\ln x)^2 - 2(x\ln x - x) + C$$

The last line uses a result that we proved a few examples ago.

Now let's try a definite integral using integration by parts:

$$\int_0^1 te^{-t} \, dt$$

$$f = t \qquad\qquad\qquad g' = e^{-t}$$
$$f' = 1 \qquad\qquad\qquad g = -e^{-t}$$

We first calculate the indefinite integral, then go back and substitute in the limits.

$$\int te^{-t} \, dt = -te^{-t} + \int e^{-t} \, dt$$

$$= -te^{-t} - e^{-t} + C$$

Therefore:

$$\int_0^1 te^{-t} \, dt = \left[-te^{-t} - e^{-t} \right]_0^1$$

$$= \left(-1e^{-1} - e^{-1} \right) - \left(0 - e^{-0} \right)$$

$$= 1 - 2e^{-1}$$

$$= 1 - \frac{2}{e}$$

You should now make sure you can do the integration practice questions.

An example of an economic application of integrals:

One simple application is to find a 'total' quantity from a 'marginal' quantity. Suppose a firm has a marginal cost $C'(x) = 1 + 2e^{\frac{x}{3}}$ where x denotes output. Then total cost is:

$$C(x) = \int (1 + 2e^{\frac{x}{3}}) \, dx = x + 6e^{\frac{x}{3}} + B, \text{ where B is the constant of integration.}$$