# Mathematics for Economists

## Optimisation

# Section 3: Optimisation

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We're now going to turn our attention to optimisation problems. Optimisation problems are central to economics because they allow us to model choice. After all, economics is all about making choices in order to achieve certain goals, whether it be an individual maximising their utility, or a firm maximising its profits.

When we formulate an optimisation problem, the first thing we need to do is specify an objective function, where the dependent variable tells us what we are trying to maximise, and the independent variables represent those variables over which we have a choice, and whose values will ultimately affect the dependent variable. The whole point of optimisation is to find a set of independent variables (or choice variables) that allow us to achieve the desired value of our objective function.

We begin by introducing the concept of the second derivative.

## 1 The second derivative

If $f$ is a differentiable function, then $f'$ is a function. If $f'$ is differentiable, we denote its derivative by $f''$, and call it the *second derivative* of $f$. This generalises to higher order derivatives:

$$\begin{aligned} f(x) &: \text{original function} \\ f'(x) &= \frac{dy}{dx} : \text{first derivative} \\ f''(x) &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} : \text{second derivative} \\ f^n(x) &= \frac{d^ny}{dx^n} : n^{\text{th}} \text{ order derivative} \end{aligned}$$

**Example 1**

$$\begin{aligned} \textit{If } y &= f(x) = x^3 + 2x^2 + 3x - 1 \\ \textit{then } \frac{dy}{dx} &= f'(x) = 3x^2 + 4x + 3 \\ \textit{and } \frac{d^2y}{dx^2} &= f''(x) = 6x + 4 \end{aligned}$$

What do these derivatives measure?

- $f'(x)$ measures the slope (rate of change) of $f(x)$
- $f''(x)$ measures the rate of change of the slope of $f(x)$

Recall that

$$\left.\begin{matrix} f'(x) > 0 \\ f'(x) < 0 \end{matrix}\right\} \text{ means that the value of the function, i.e. } f(x) \text{, is } \left\{\begin{matrix} \text{increasing} \\ \text{decreasing} \end{matrix}\right.$$

Now

$$\left.\begin{matrix} f''(x) > 0 \\ f''(x) < 0 \end{matrix}\right\} \text{ means that the slope of the curve, i.e. } f'(x) \text{, is } \left\{\begin{matrix} \text{increasing} \\ \text{decreasing} \end{matrix}\right.$$

Thus for positive first derivatives:

- $f'(x) > 0$ and $f''(x) > 0 \Rightarrow$ the slope of the curve is positive and increasing, so $f(x)$ is increasing at an increasing rate.
- $f'(x) > 0$ and $f''(x) < 0 \Rightarrow$ the slope of the curve is positive but decreasing, $f(x)$ is increasing at a decreasing rate.

The case of a negative first derivative can be interpreted analogously, but we must be a little more careful:

- $f'(x) < 0$ and $f''(x) > 0 \Rightarrow$ the slope of the curve is negative and increasing, but this means the slope is changing from $(-11)$ to a larger number $(-10)$. In other words, the negative slope becomes *less* steep as $x$ increases.
- $f'(x) < 0$ and $f''(x) < 0 \Rightarrow$ the slope of the curve is negative and decreasing, and so the negative slope becomes *steeper* as $x$ increases.

**Definition 1** *If we pick any pair of points $M$ and $N$ on the curve $f(x)$, join them by a straight line and the line segment $MN$ lies entirely* below *the curve, then we say $f(x)$ is a* strictly concave function*.*

*If the line segment $MN$ lies either below the curve, or along (coinciding with) the curve, then $f(x)$ is a* concave function *(but not strictly so).*

**Definition 2** *If we pick any pair of points $M$ and $N$ on the curve $f(x)$, join them by a straight line and the line segment $MN$ lies entirely* above *the curve, then we say $f(x)$ is a* strictly convex function*.*

*If the line segment $MN$ lies either above the curve, or along (coinciding with) the curve, then $f(x)$ is a* convex function *(but not strictly so).*

Note that (nonstrictly) concave or convex functions may contain linear segments, but strictly concave or convex functions can never contain a linear segment anywhere. Strictly concave (convex) functions must be concave (convex) functions, but the converse is not true.

The second derivative relates to the curvature of $f(x)$ as follows:

- if $f''(x) < 0$ for all $x$, then $f(x)$ must be a strictly concave function.
- if $f''(x) > 0$ for all $x$, then $f(x)$ must be a strictly convex function.

## 2 Finding and classifying critical points

We begin with the simple case where $y$ is a function of one variable.

Consider a function $y = f(x)$. To find and classify its critical points, we:

1. Find $f'(x)$ and $f''(x)$.
2. Find the critical points by setting $f'(x) = 0$.
3. Classify the critical points using the second derivative test:

   For each critical point $(x^*, y^*)$, calculate $f''(x^*)$. If

   (a) $f''(x^*) < 0$, then $(x^*, y^*)$ is a maximum.

   (b) $f''(x^*) > 0$, then $(x^*, y^*)$ is a minimum.

   (c) $f''(x^*) = 0$, then $(x^*, y^*)$ may be a maximum, minimum or point of inflexion. We must then use the following test: Find the sign of $f'(x)$ when $x$ is slightly less than $x^*$ (which is written $x = x^*-$) and when $x$ is slightly greater than $x^*$ (which is written $x = x^*+$). If

   - $f'(x)$ changes from positive to negative when $x$ changes from $x^*-$ to $x^*+$, then $(x^*, y^*)$ is a maximum.
   - $f'(x)$ changes from negative to positive when $x$ changes from $x^*-$ to $x^*+$, then $(x^*, y^*)$ is a minimum.
   - $f'(x)$ does not change sign, then $(x^*, y^*)$ is a critical point of inflexion.

**Example 2** *Find and classify the critical points of the function*

$$y = 2x^3 - 3x^2 - 12x + 9$$

*Step 1:*

$$\frac{dy}{dx} = 6x^2 - 6x - 12$$
$$\frac{d^2y}{dx^2} = 6(2x - 1)$$

*Step 2:*

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ \Rightarrow 6x^2 - 6x - 12 &= 0 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ \Rightarrow x &= -1; 2 \end{aligned}$$

*The critical points are therefore* $(-1, 16)$ *and* $(2, -11)$.
*Step 3:*

$$\begin{aligned} x &= -1 : \frac{d^2y}{dx^2} = -18 < 0 \Rightarrow \text{ this is a relative maximum} \\ x &= 2 : \frac{d^2y}{dx^2} = 18 > 0 \Rightarrow \text{ this is a relative minimum} \end{aligned}$$

**Example 3** *Find and classify the critical points of the function*

$$y = x^3 - 3x^2 + 2$$

*Step 1:*

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 - 6x \\ \frac{d^2y}{dx^2} &= 6x - 6 \end{aligned}$$

*Step 2:*

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ \Rightarrow 3x^2 - 6x &= 0 \\ 3x(x-2) &= 0 \\ \Rightarrow x &= 0 \text{ or } x = 2 \end{aligned}$$

*The critical points are therefore* $(0, 2)$ *and* $(2, -2)$.
*Step 3:*

$$\begin{aligned} x &= 0 : \frac{d^2y}{dx^2} = -6 < 0 \Rightarrow \text{ this is a local maximum} \\ x &= 2 : \frac{d^2y}{dx^2} = 6 > 0 \Rightarrow \text{ this is a local minimum} \end{aligned}$$

**Example 4** *Find and classify the extremum of the function*

$$y = 4x^2 - x$$

*Step 1:*

$$\begin{aligned}\frac{dy}{dx} &= 8x - 1 \\ \frac{d^2y}{dx^2} &= 8\end{aligned}$$

*Step 2:*

$$\begin{aligned}\frac{dy}{dx} &= 0 \\ \Rightarrow 8x - 1 &= 0 \\ \Rightarrow x &= \frac{1}{8}\end{aligned}$$

*The extremum is therefore* $\left(\frac{1}{8}, -\frac{1}{16}\right)$.

*Step 3:*

$$x = \frac{1}{8} : \frac{d^2y}{dx^2} = 8 > 0 \Rightarrow \text{ this is a relative minimum}$$

**Example 5** The ambiguous case: *We look at the case 3 (c).*

*Consider the following three cases:*

$$\begin{aligned}(i)\ y &= x^4 \\ (ii)\ y &= -x^4 \\ (iii)\ y &= x^3\end{aligned}$$

*In each case, there is a critical point at the origin* $(0,0)$, *and* $\frac{d^2y}{dx^2} = 0$ *when* $x = 0$.

*However, (* you should verify this*) the critical point is a minimum in case (i), a maximum in case (ii) and point of inflexion in case (iii).*

## 2.1 Economic Application

**Example 6** *Find and classify the extremum of the average cost function*

$$AC = f(Q) = Q^2 - 5Q + 8$$

*Step 1:*

$$\begin{aligned}f'(Q) &= 2Q - 5 \\ f''(Q) &= 2\end{aligned}$$

*Step 2:*

$$\begin{aligned} f'(Q) &= 0 \\ \Rightarrow 2Q - 5 &= 0 \\ \Rightarrow Q &= \frac{5}{2} \end{aligned}$$

*When* $Q = \frac{5}{2}, AC = \frac{7}{4}$

*Step 3:*

$$Q = \frac{5}{2} : f''(Q) = 2 > 0 \Rightarrow \textit{ this is a relative minimum}$$

*This accords with our economic intuition, since the average cost curve is U-shaped.*

# 3 Optimisation

Optimisation is concerned with finding the maximum or minimum value of a function usually, but not always, subject to some constraint(s) on the independent variable(s). Whether the maximum or minimum is required depends on whether the function represents a desirable quantity such as profit or an undesirable quantity such as cost.

## 3.1 Maxima and minima: local and global

A maximum point is often called a *local maximum* to emphasize the fact that $y$ has its greatest value in a neighbourhood of the maximum point. Similarly a minimum point is often called a *local minimum*.

### 3.1.1 Conditions for local optima

Consider the twice differentiable function $f(x)$.

1. If $f'(x^*) = 0$ and $f''(x^*) < 0$, the function $f$ has a local maximum where $x = x^*$.
2. If $f'(x^*) = 0$ and $f''(x^*) > 0$, the function $f$ has a local minimum where $x = x^*$.

We refer to the condition $f'(x^*) = 0$ as the *first-order condition* (FOC), and to the conditions $f''(x^*) < 0$ and $f''(x^*) > 0$ as *second-order conditions* (SOCs).

### 3.1.2 Local vs global

A global maximum point is a local maximum point $(x^*, y^*)$ of the curve $y = f(x)$ which has the additional property that

$$y^* \geq f(x), \forall x \in \mathbb{R}$$

Similarly, a global minimum point is a local minimum point $(x^*, y^*)$ of the curve $y = f(x)$ which has the additional property that

$$y^* \leq f(x), \forall x \in \mathbb{R}$$

Global maxima (minima) may be found as follows: Start by finding the local maxima (minima). *Provided a global maximum (minimum) exists*, it (or they) may be found by comparing the values of $y$ at the local maxima (minima). In general, the only way to tell whether a global maximum (minimum) exists is to sketch the curve, and in particular to consider how $y$ behaves as $x \to \pm\infty$.

## 3.2 Economic Application

Consider the problem of the profit maximising firm:

$$\max_Q \ \Pi(Q) = R(Q) - C(Q)$$

where $Q$ denotes output, $R$ revenue, $C$ cost and $\Pi$ profit.
FOC:

$$\begin{aligned} \Pi'(Q) = R'(Q) - C'(Q) &= 0 \\ \Rightarrow R'(Q) &= C'(Q) \\ MR &= MC \end{aligned}$$

SOC

$$\begin{aligned} \Pi''(Q) &= R''(Q) - C''(Q) \\ \text{If } R''(Q) &< C''(Q), \text{ then } \Pi''(Q) < 0 \end{aligned}$$

This says that the slope of the marginal revenue curve must be less than the slope of the marginal cost curve at the profit maximising level of output.

**Example 7** *Suppose a firm faces the following total revenue and total cost functions*

$$\begin{aligned} R(Q) &= 1200Q - 2Q^2 \\ C(Q) &= Q^3 - 61.25Q^2 + 1528.5Q + 2000 \end{aligned}$$

*Find the maximum profit that can be attained.*

$$\begin{aligned}\Pi(Q) &= R(Q) - C(Q) \\ &= 1200Q - 2Q^2 - \left(Q^3 - 61.25Q^2 + 1528.5Q + 2000\right) \\ &= -Q^3 + 59.25Q^2 - 328.5Q - 2000\end{aligned}$$

*The problem is then to*

$$\max_{Q} \ \Pi(Q) = -Q^3 + 59.25Q^2 - 328.5Q - 2000$$

*Find the first and second derivatives*

$$\begin{aligned}\Pi'(Q) &= -3Q^2 + 118.5Q - 328.5 \\ \Pi''(Q) &= -6Q + 118.5\end{aligned}$$

*FOC*

$$\begin{aligned}\Pi'(Q) = -3Q^2 + 118.5Q - 328.5 &= 0 \\ \Rightarrow Q = 3 \ \textit{or} \ Q &= 36.5 \ \textit{(use the quadratic formula)}\end{aligned}$$

*SOC*

$$\begin{aligned}\Pi''(3) &= 100.5 > 0 \Rightarrow \ \textit{this a relative minimum} \\ \Pi''(36.5) &= -100.5 < 0 \Rightarrow \ \textit{this a relative maximum}\end{aligned}$$

*Therefore, to maximise profits the firm should produce* 36.5 *units of output. When* $Q = 36.5, \Pi = 16318.44$.

**Example 8** *Suppose a monopolist faces the demand function*

$$x = 100 - p$$

*where* $x$ *is output and* $p$ *is price, and let the monopolist's total cost be*

$$C(x) = \frac{1}{3}x^3 - 7x^2 + 111x + 50$$

*Find the output and price that maximise profit, and the maximal profit.*

*We begin by solving the demand equation for price in terms of output, and multiplying by output to obtain the revenue:*

$$R(x) = px = (100 - x)\,x = 100x - x^2$$

*Profit is given by*

$$\begin{aligned}\Pi(x) &= R(x) - C(x) \\ &= -\frac{1}{3}x^3 + 6x^2 - 11x - 50\end{aligned}$$

*By differentiation:*

$$\begin{aligned}\Pi'(x) &= -x^2 + 12x - 11 \\ \Pi''(x) &= -2x + 12\end{aligned}$$

*FOC:*

$$\begin{aligned}\Pi'(x) &= 0 \\ \Rightarrow -x^2 + 12x - 11 &= 0 \\ (x-1)(x-11) &= 0 \\ x &= 1; 11\end{aligned}$$

*SOCs:*

$$\begin{aligned}\Pi''(1) &= 10 > 0 \Rightarrow \textit{ this is a local minimum} \\ \Pi''(11) &= -10 < 0 \Rightarrow \textit{ this is a local maximum}\end{aligned}$$

*Thus profit is maximised at output $x = 11$ and price $p = 89$, and the maximal profit is* 111.33.

We now consider the case of the function $z = f(x, y)$. We first consider the partial derivatives of this function.

# 4 Partial derivatives

You will recall that we can define the following partial derivatives:

$$\begin{aligned}f_x &= \frac{\partial f}{\partial x} \\ f_y &= \frac{\partial f}{\partial y}\end{aligned}$$

We can differentiate these again:

$$\begin{aligned}f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\ f_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\end{aligned}$$

These are known as the *second partial derivatives* of the function $f(x, y)$.

Similarly we may define the expressions

$$\begin{aligned} f_{xy} &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\ f_{yx} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \end{aligned}$$

These are called the *cross (or mixed) partial derivatives* of $f(x, y)$.

If the two cross partial derivatives are continuous, then according to Young's theorem they will be equal, i.e.

$$f_{xy} = f_{yx}$$

**Example 9** *Consider the function* $f(x, y) = x^2 y + y^5$.

$$\begin{aligned} f_x &= 2xy \\ f_y &= x^2 + 5y^4 \\ f_{xx} &= 2y \\ f_{yy} &= 20y^3 \\ f_{xy} &= f_{yx} = 2x \end{aligned}$$

**Example 10** *If* $f(x, y) = x^3 + 5xy - y^2$, *then*

$$\begin{aligned} f_x &= 3x^2 + 5y \\ f_y &= 5x - 2y \\ f_{xx} &= 6x \\ f_{yy} &= -2 \\ f_{xy} &= f_{yx} = 5 \end{aligned}$$

**Example 11** *If* $f(x, y) = 4x^2 - 3xy + 2y^2$, *then*

$$\begin{aligned} f_x &= 8x - 3y \\ f_y &= -3x + 4y \\ f_{xx} &= 8 \\ f_{yy} &= 4 \\ f_{xy} &= f_{yx} = -3 \end{aligned}$$

**Example 12** *If* $f(x,y) = 10x^3 + 15xy - 6y^2$, *then*

$$\begin{aligned} f_x &= 30x^2 + 15y \\ f_y &= 15x - 12y \\ f_{xx} &= 60x \\ f_{yy} &= -12 \\ f_{xy} &= f_{yx} = 15 \end{aligned}$$

### 4.1 Gradient and Hessian

Given the smooth function $f(x,y)$, we may define the *gradient vector*

$$Df(x,y) = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

and the *Hessian matrix*

$$\mathbf{H} = D^2 f(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

By Young's theorem, $f_{xy} = f_{yx}$ and so the Hessian matrix is symmetric.

## 5 Optimisation with several variables

Recall that for a function of one variable $f(x)$, the conditions for a local maximum and minimum were:

1. If $f'(x^*) = 0$ and $f''(x^*) < 0$, the function $f$ has a local maximum where $x = x^*$.
2. If $f'(x^*) = 0$ and $f''(x^*) > 0$, the function $f$ has a local minimum where $x = x^*$.

How can we extend these results to functions of two variables?

The two-variable analogue of the first derivative is the gradient vector, and so statements about the first derivative being zero translate into statements about gradient vectors being zero-vectors. The two-variable analogue of the second derivative is the Hessian, which is a symmetric matrix, and so statements about second derivatives being negative translate into statements about Hessians being negative definite, and statements about second derivatives being positive translate into statements about Hessians being positive definite.

Thus, given a function $f(x,y)$,the conditions for a local maximum and minimum are:

1. If $Df(x^*,y^*) = \mathbf{0}$ and $\mathbf{H} = D^2 f(x^*,y^*)$ is a negative definite symmetric matrix, then the function $f(x,y)$ has a local maximum at $(x^*,y^*)$.

2. If $Df(x^*, y^*) = \mathbf{0}$ and $\mathbf{H} = D^2 f(x^*, y^*)$ is a positive definite symmetric matrix, then the function $f(x, y)$ has a local minimum at $(x^*, y^*)$.

**Example 13** *Find and classify the extrema of the function*

$$f(x, y) = x^2 + xy + 2y^2 + 3$$

*We calculate the gradient and Hessian:*

$$\begin{aligned} Df(x, y) &= \begin{bmatrix} f_x \\ f_y \end{bmatrix} \\ &= \begin{bmatrix} 2x + y \\ x + 4y \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{H} &= D^2 f(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \end{aligned}$$

*FOCs:*

$$\begin{aligned} Df(x, y) &= \mathbf{0} \\ \Rightarrow \begin{bmatrix} 2x + y \\ x + 4y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

*Solve these simultaneously to get $x = 0, y = 0$.*
*SOC:*

$$\begin{aligned} |\mathbf{H}_1| &= |2| = 2 > 0 \\ |\mathbf{H}_2| &= |\mathbf{H}| = 7 > 0 \end{aligned}$$

*Thus, **H** is positive definite and so $x = 0, y = 0, f(x, y) = 3$ is a local minimum.*

**Example 14** *Find and classify the critical points of the function*

$$f(x, y) = -5x^2 - y^2 + 2xy + 6x + 2y + 7$$

*We calculate the gradient and Hessian:*

$$\begin{aligned} Df(x, y) &= \begin{bmatrix} f_x \\ f_y \end{bmatrix} \\ &= \begin{bmatrix} -10x + 2y + 6 \\ -2y + 2x + 2 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
\mathbf{H} &= D^2 f(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \\
&= \begin{bmatrix} -10 & 2 \\ 2 & -2 \end{bmatrix}
\end{aligned}
$$

*FOCs:*

$$
\begin{aligned}
Df(x,y) &= \mathbf{0} \\
\Rightarrow \begin{bmatrix} -10x + 2y + 6 \\ -2y + 2x + 2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

*Solve these simultaneously to get* $x = 1, y = 2$.
*SOC:*

$$
\begin{aligned}
|\mathbf{H}_1| &= |-10| = -10 < 0 \\
|\mathbf{H}_2| &= |\mathbf{H}| = 20 - 4 = 16 > 0
\end{aligned}
$$

*Thus,* $\mathbf{H}$ *is negative definite and so* $f(1,2) = 12$ *is a local maximum.*

**Example 15** *Find and classify the extrema of the function*

$$
f(x,y) = ax^2 + by^2 + c
$$

*We calculate the gradient and Hessian:*

$$
\begin{aligned}
Df(x,y) &= \begin{bmatrix} f_x \\ f_y \end{bmatrix} \\
&= \begin{bmatrix} 2ax \\ 2by \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{H} &= D^2 f(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \\
&= \begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix}
\end{aligned}
$$

*FOCs:*

$$
\begin{aligned}
Df(x,y) &= \mathbf{0} \\
\Rightarrow \begin{bmatrix} 2ax \\ 2by \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

*Solve these to get $x = 0, y = 0$.*
*SOC:*

$$\begin{aligned} |\mathbf{H}_1| &= |2a| = 2a \\ |\mathbf{H}_2| &= |\mathbf{H}| = 4ab \end{aligned}$$

*The definiteness of* $\mathbf{H}$ *depends on the values of a and b, so let's look at the different possibilities:*

1. $a > 0, b > 0 : |\mathbf{H}_1| > 0, |\mathbf{H}_2| > 0 \Rightarrow \mathbf{H}$ *is positive definite and so* $f(0,0)$ *is a relative minimum.*
2. $a < 0, b < 0 : |\mathbf{H}_1| < 0, |\mathbf{H}_2| > 0 \Rightarrow \mathbf{H}$ *is negative definite and so* $f(0,0)$ *is a relative maximum.*
3. $a > 0, b < 0 : |\mathbf{H}_1| > 0, |\mathbf{H}_2| < 0 \Rightarrow \mathbf{H}$ *is indefinite.*
4. $a < 0, b > 0 : |\mathbf{H}_1| < 0, |\mathbf{H}_2| < 0 \Rightarrow \mathbf{H}$ *is indefinite.*

## 5.1 Higher dimensions

We can generalise these results to functions of $n$ variables. The gradient vector will be an $n$-vector and the Hessian matrix will be an $n \times n$ matrix.

Thus, given a function $f(\mathbf{x})$,the conditions for a local maximum and minimum are:

1. If $Df(\mathbf{x}^*) = \mathbf{0}$ and $\mathbf{H} = D^2 f(\mathbf{x}^*)$ is a negative definite symmetric matrix, then the function $f(\mathbf{x})$ has a local maximum at $\mathbf{x} = \mathbf{x}^*$.
2. If $Df(\mathbf{x}^*) = \mathbf{0}$ and $\mathbf{H} = D^2 f(\mathbf{x}^*)$ is a positive definite symmetric matrix, then the function $f(\mathbf{x})$ has a local minimum at $\mathbf{x} = \mathbf{x}^*$.

## 5.2 Global optima

Given a function $f(\mathbf{x})$,the conditions for a unique global maximum and minimum are:

1. If $Df(\mathbf{x}^*) = \mathbf{0}$ and $\mathbf{H} = D^2 f(\mathbf{x})$ is everywhere negative definite, then the function $f(\mathbf{x})$ is strictly concave and has a unique global maximum at $\mathbf{x} = \mathbf{x}^*$.
2. If $Df(\mathbf{x}^*) = \mathbf{0}$ and $\mathbf{H} = D^2 f(\mathbf{x})$ is everywhere positive definite, then the function $f(\mathbf{x})$ is strictly convex and has a unique global minimum at $\mathbf{x} = \mathbf{x}^*$.

Given a function $f(\mathbf{x})$,the conditions for a global maximum and minimum are:

1. If $Df(\mathbf{x}^*) = \mathbf{0}$ and $\mathbf{H} = D^2f(\mathbf{x})$ is everywhere negative semidefinite, then the function $f(\mathbf{x})$ is concave and has a global maximum at $\mathbf{x} = \mathbf{x}^*$.

2. If $Df(\mathbf{x}^*) = \mathbf{0}$ and $\mathbf{H} = D^2f(\mathbf{x})$ is everywhere positive semidefinite, then the function $f(\mathbf{x})$ is convex and has a global minimum at $\mathbf{x} = \mathbf{x}^*$.

**Example 16** *In the last three examples, the Hessian matrices were everywhere negative or positive definite and thus the extrema found are in fact unique global maxima or minima.*

**Example 17** *Find and classify the extrema of the function*

$$f(x, y) = 8x^3 + 2xy - 3x^2 + y^2 + 1$$

*We calculate the gradient and Hessian:*

$$\begin{aligned} Df(x,y) &= \begin{bmatrix} f_x \\ f_y \end{bmatrix} \\ &= \begin{bmatrix} 24x^2 + 2y - 6x \\ 2x + 2y \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{H} &= D^2f(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \\ &= \begin{bmatrix} 48x - 6 & 2 \\ 2 & 2 \end{bmatrix} \end{aligned}$$

*FOCs:*

$$\begin{aligned} Df(x,y) &= \mathbf{0} \\ \Rightarrow \begin{bmatrix} 24x^2 + 2y - 6x \\ 2x + 2y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

*Solve these simultaneously:*

$$\begin{aligned} 24x^2 + 2y - 6x &= 0 = 2x + 2y \\ \Rightarrow 24x^2 + 2y - 6x &= 2x + 2y \\ 24x^2 - 8x &= 0 \\ x(24x - 8) &= 0 \\ \therefore x &= 0 \text{ or } x = \frac{8}{24} = \frac{1}{3} \end{aligned}$$

*SOC:*

$$|\mathbf{H}_1| = |48x - 6| = 48x - 6 \quad < \quad 0 \text{ when } x < \frac{6}{48}$$
$$> \quad 0 \text{ when } x > \frac{6}{48}$$

*Thus, the Hessian cannot be everywhere positive or negative definite.*
*We evaluate the Hessian at each critical point.*
*When* $x = 0$ :

$$|\mathbf{H}_1| = |-6| = -10 < 0$$
$$|\mathbf{H}_2| = |\mathbf{H}| = -16 < 0$$

*Thus,* $\mathbf{H}$ *is neither negative nor positive definite and so* $x = 0$ *is neither a local maximum nor a local minimum.*
*When* $x = \frac{1}{3}$ :

$$|\mathbf{H}_1| = |10| = 10 > 0$$
$$|\mathbf{H}_2| = |\mathbf{H}| = 16 > 0$$

*Thus,* $\mathbf{H}$ *is positive definite and so* $x = \frac{1}{3}$ *is a local minimum.*
*When* $x = \frac{1}{3}, y = -x = -\frac{1}{3}$ *and* $f(x, y) = -\frac{23}{27}$.

**Example 18** *Find and classify the extrema of the function*

$$f(x, y) = x^3 + y^2 - 4xy - 3x$$

*Gradient and Hessian:*

$$Df(x, y) = \begin{bmatrix} 3x^2 - 4y - 3 \\ 2y - 4x \end{bmatrix}$$

$$\mathbf{H} = D^2 f(x, y) = \begin{bmatrix} 6x & -4 \\ -4 & 2 \end{bmatrix}$$

*FOCs:*

$$Df(x, y) = \mathbf{0}$$
$$\Rightarrow \begin{bmatrix} 3x^2 - 4y - 3 \\ 2y - 4x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

*Solve these simultaneously to get the critical points:*

*(a)* $x = 3, y = 6, f(3,6) = -18$

*(b)* $x = -\frac{1}{3}, y = -\frac{2}{3}, f\left(-\frac{1}{3}, -\frac{2}{3}\right) = \frac{14}{27}$

*SOC:*

$$\begin{aligned} |\mathbf{H}_1| = |6x| = 6x &< 0 \text{ when } x < 0 \\ &> 0 \text{ when } x > 0 \end{aligned}$$

*Thus, the Hessian cannot be everywhere positive or negative definite.*
*We evaluate the Hessian at each critical point. Consider* $x = 3, y = 6$ *:*

$$\begin{aligned} \mathbf{H} &= D^2 f(3,6) \\ &= \begin{bmatrix} 18 & -4 \\ -4 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} |\mathbf{H}_1| &= |18| = 18 > 0 \\ |\mathbf{H}_2| &= |\mathbf{H}| = 20 > 0 \end{aligned}$$

*Thus,* $\mathbf{H}$ *is positive definite at* $x = 3, y = 6$ *and so* $f(3,6)$ *is a local minimum.*
*Consider* $x = -\frac{1}{3}, y = -\frac{2}{3}$*:*

$$\begin{aligned} \mathbf{H} &= D^2 f\left(-\frac{1}{3}, -\frac{2}{3}\right) \\ &= \begin{bmatrix} -2 & -4 \\ -4 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} |\mathbf{H}_1| &= |-2| = -2 < 0 \\ |\mathbf{H}_2| &= |\mathbf{H}| = -20 < 0 \end{aligned}$$

*Thus,* $\mathbf{H}$ *is neither positive nor negative definite at* $x = -\frac{1}{3}, y = -\frac{2}{3}$ *and so* $f\left(-\frac{1}{3}, -\frac{2}{3}\right)$ *is neither a local minimum nor maximum..*

## 5.3 Economic Application

**Example 19** *A firm produces two products, with price* $P_1$ *and* $P_2$ *respectively. The firm's total revenue function is given by*

$$TR = P_1 Q_1 + P_2 Q_2$$

*The firm's total cost function is given by*

$$TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$$

*Find the profit maximising output of each good for this firm under perfect competition. First, we need to specify the objective function. The firm wants to maximise profits:*

$$\max_{Q_1,Q_2} \; \Pi = TR - TC = P_1Q_1 + P_2Q_2 - 2Q_1^2 - Q_1Q_2 - 2Q_2^2$$

*Note that under perfect competition firms are price takers, so the choice variables for the firm must be the quantities of each good that it produces.*
*Gradient and Hessian:*

$$\begin{aligned} D\Pi(x,y) &= \begin{bmatrix} \Pi_{Q_1} \\ \Pi_{Q_2} \end{bmatrix} \\ &= \begin{bmatrix} P_1 - 4Q_1 - Q_2 \\ P_2 - Q_1 - 4Q_2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{H} &= D^2\Pi(x,y) \\ &= \begin{bmatrix} \Pi_{Q_1Q_1} & \Pi_{Q_1Q_2} \\ \Pi_{Q_2Q_1} & \Pi_{Q_2Q_2} \end{bmatrix} \\ &= \begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix} \end{aligned}$$

*FOCs:*

$$\begin{aligned} D\Pi(x,y) &= \mathbf{0} \\ \Rightarrow \begin{bmatrix} P_1 - 4Q_1 - Q_2 \\ P_2 - Q_1 - 4Q_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

*Solve these simultaneously to get* $Q_1^* = \dfrac{4P_1 - P_2}{15}, Q_2^* = \dfrac{4P_2 - P_1}{15}$.
*SOC:*

$$\begin{aligned} |\mathbf{H}_1| &= |-4| = -4 < 0 \\ |\mathbf{H}_2| &= |\mathbf{H}| = 15 > 0 \end{aligned}$$

*Thus,* $\mathbf{H}$ *is everywhere negative definite and so the point is a unique global maximum. The firm will be maximising its profits when it produces* $Q_1^* = \dfrac{4P_1 - P_2}{15}, Q_2^* = \dfrac{4P_2 - P_1}{15}$.

**Example 20** *Consider a monopolist producing two goods $X$ and $Y$. Let the quantities produced of the two goods be $x$ and $y$, and let the prices charged be $p_X$ and $p_Y$. The inverse demand functions for the goods are:*

$$\begin{aligned} p_X &= \frac{1}{10}(54 - 3x - y) \\ p_Y &= \frac{1}{5}(48 - x - 2y) \end{aligned}$$

*and the firm's total cost is*

$$C(x, y) = 8 + 1.5x + 1.8y$$

*Hence revenue is*

$$\begin{aligned} R(x, y) &= p_X x + p_Y y \\ &= \frac{1}{10}\left(54x + 96y - 3x^2 - 3xy - 4y^2\right) \end{aligned}$$

*and profit is*

$$\begin{aligned} \Pi(x, y) &= R(x, y) - C(x, y) \\ &= \frac{1}{10}\left(-80 + 39x + 78y - 3x^2 - 3xy - 4y^2\right) \end{aligned}$$

*Gradient and Hessian:*

$$\begin{aligned} D\Pi(x, y) &= \begin{bmatrix} \Pi_x \\ \Pi_y \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{10}(39 - 6x - 3y) \\ \frac{1}{10}(78 - 3x - 8y) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{H} &= D^2\Pi(x, y) \\ &= \begin{bmatrix} \Pi_{xx} & \Pi_{xy} \\ \Pi_{yx} & \Pi_{yy} \end{bmatrix} \\ &= \begin{bmatrix} -0.6 & -0.3 \\ -0.3 & -0.8 \end{bmatrix} \end{aligned}$$

*FOCs:*

$$\begin{aligned} D\Pi(x, y) &= \mathbf{0} \\ \Rightarrow \begin{bmatrix} \frac{1}{10}(39 - 6x - 3y) \\ \frac{1}{10}(78 - 3x - 8y) \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

*Solve these simultaneously to get* $x = 2, y = 9$*. Then* $p_X = 3.9, p_Y = 5.6$ *and* $\Pi = 31$*.*
*SOC:*

$$\begin{aligned} |\mathbf{H}_1| &= |-0.6| = -0.6 < 0 \\ |\mathbf{H}_2| &= |\mathbf{H}| = 0.39 > 0 \end{aligned}$$

*Thus,* $\mathbf{H}$ *is everywhere negative definite and so* $\Pi(2, 9) = 31$ *is a unique global maximum.*

**Example 21** *As financial advisor to* The Journal of Important Stuff*, you need to determine how many pages should be allocated to important stuff about economics (E) and how much should be allocated to other unimportant issues (U) in order to maximise your sales. Determine how many pages you should allocate to economics if your sales function is given by*

$$S(U, E) = 100U + 310E - \frac{1}{2}U^2 - 2E^2 - UE$$

*Our problem is to*

$$\max_{U,E} \; S = 100U + 310E - \frac{1}{2}U^2 - 2E^2 - UE$$

*Gradient and Hessian:*

$$\begin{aligned} D\Pi(x, y) &= \begin{bmatrix} \partial S/\partial U \\ \partial S/\partial E \end{bmatrix} \\ &= \begin{bmatrix} 100 - U - E \\ 310 - 4E - U \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{H} &= D^2\Pi(x, y) \\ &= \begin{bmatrix} S_{UU} & S_{UE} \\ S_{EU} & S_{EE} \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 \\ -1 & -4 \end{bmatrix} \end{aligned}$$

*FOCs:*

$$\begin{aligned} D\Pi(x, y) &= \mathbf{0} \\ \Rightarrow \begin{bmatrix} 100 - U - E \\ 100 - U - E \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

*Solve these simultaneously to get* $U = 30, E = 70$*.*
*SOC:*

$$\begin{aligned} |\mathbf{H}_1| &= |-1| = -1 < 0 \\ |\mathbf{H}_2| &= |\mathbf{H}| = 3 > 0 \end{aligned}$$

*Thus,* $\mathbf{H}$ *is everywhere negative definite and so* $S(30, 70)$ *is a unique global maximum.*

# 6 Constrained Optimisation

In the optimisation problems we have studied up until now, we have assumed that the firm or individual is able to make their choice freely, without facing any constraints. In reality, however, our choices are often constrained by our budgets, our time availability, and so on. In order to incorporate this element of decision-making into our analysis, we undertake constrained optimisation. Here, we still aim to maximize some objective function, but in so doing, have to take some constraint into account as well.

To do this, we use the Lagrangian multiplier method.

## 6.1 The Lagrangian

Let $f = (x, y)$ and $g = (x, y)$ be functions of two variables. Our problem is to

$$\max_{x,y} f(x,y) \text{ subject to the constraint } g(x,y) = c$$

To solve this problem, we employ the Lagrangian

$$L(x,y,\lambda) = f(x,y) + \lambda\left(c - g(x,y)\right)$$

Here, the symbol $\lambda$ represents the Lagrangian multiplier. (Note that we always write the Lagrangian function in terms of the *constant minus the choice variables*) Effectively, we are incorporating the constraint into the function, and treating $\lambda$ as an additional variable in the function.

The first-order conditions are:

$$\begin{aligned} \frac{\partial L}{\partial x} &= 0 \\ \frac{\partial L}{\partial y} &= 0 \\ \frac{\partial L}{\partial \lambda} &= 0 \end{aligned}$$

We solve these three equations simultaneously to find the optimal $x^*, y^*, \lambda^*$.

**Example 22** *The problem is to*

$$\max_{x,y} 4xy - 2x^2 + y^2 \text{ subject to } 3x + y = 5$$

*Set up the Lagrangian*

$$L(x,y,\lambda) = 4xy - 2x^2 + y^2 + \lambda\left(5 - 3x - y\right)$$

*FOCs:*

$$\frac{\partial L}{\partial x} = 4y - 4x - 3\lambda = 0$$
$$\frac{\partial L}{\partial y} = 4x + 2y - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 5 - 3x - y = 0$$

*Solve simultaneously to give* $x^* = -1, y^* = 8, \lambda^* = 12$.

In order to confirm that this represents a constrained maximum, we need to consider the consider the second order conditions.

## 6.2 Second-order conditions

We first define the bordered Hessian. The bordered Hessian consists of the Hessian $\begin{bmatrix} L_{xx} & L_{xy} \\ L_{yx} & L_{yy} \end{bmatrix}$ bordered on top and to the left by the derivatives of the constraint, $g_x$ and $g_y$, plus a zero in the principal diagonal. The bordered Hessian is symmetric, and is denoted by $\overline{\mathbf{H}}$, where the bar on top symbolises the border.

$$\overline{\mathbf{H}} = \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix}$$

The first value in the top left-hand corner of the bordered Hessian will always be zero. The remaining values in the border are simply given by the derivative of the constraint.

We will evaluate the leading principal minors of the bordered Hessian to check whether our solution constitutes a minimum or maximum point. However, the conditions here are somewhat different, so be careful.

The first important point is that when we identify the leading principal minors of a bordered Hessian, we ignore the border to figure out what rows and columns to include in the leading principal minors, but then, before we evaluate the leading principal minors, add the border back in. If this sounds confusing, an example might make it clearer.

Consider the bordered Hessian:

$$\overline{\mathbf{H}} = \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix}$$

To find the second leading principal minor $\left|\overline{\mathbf{H}}_2\right|$, we ignore the border of the Hessian, and pretend that the matrix we're looking at is given by

$$\overline{\mathbf{H}} = \begin{bmatrix} L_{xx} & L_{xy} \\ L_{yx} & L_{yy} \end{bmatrix}$$

In this case, we already have only two rows and two columns, so this determinant would be our second leading principal minor. . . . HOWEVER, we must now add the border back in before we evaluate the determinant. In effect, the second principal minor of our bordered Hessian is given by:

$$\left|\overline{\mathbf{H}}_2\right| = \left|\overline{\mathbf{H}}\right| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix}$$

Note that the first leading principal minor $\left|\overline{\mathbf{H}}_1\right|$ is always negative and so we ignore it:

$$\begin{aligned} \left|\overline{\mathbf{H}}_1\right| &= \begin{vmatrix} 0 & g_x \\ g_x & L_{xx} \end{vmatrix} \\ &= 0 - g_x^2 \\ &= -g_x^2 < 0 \end{aligned}$$

The conditions for local constrained extremum are:

1. If $L_\lambda = L_x = L_y = 0$ and $\left|\overline{\mathbf{H}}_2\right|$ is positive, then the function $f(x, y)$ has a local constrained maximum at $(x^*, y^*)$.
2. If $L_\lambda = L_x = L_y = 0$ and $\left|\overline{\mathbf{H}}_2\right|$ is negative, then the function $f(x, y)$ has a local constrained minimum at $(x^*, y^*)$.

This of course generalises to the $n$-variable case. Given $f(x_1, x_2, \ldots, x_n)$ subject to $g(x_1, x_2, \ldots, x_n) = c$; with $L = f(x_1, x_2, \ldots, x_n) + \lambda(c - g(x_1, x_2, \ldots, x_n))$ :

First we note that:

- $\overline{\mathbf{H}}$ is negative definite iff $\left|\overline{\mathbf{H}}_2\right| > 0, \left|\overline{\mathbf{H}}_3\right| < 0, \left|\overline{\mathbf{H}}_4\right| > 0$, etc. (i.e. the leading principal minors alternate in sign beginning with a positive)
- $\overline{\mathbf{H}}$ is positive definite iff. $\left|\overline{\mathbf{H}}_2\right| < 0, \left|\overline{\mathbf{H}}_3\right| < 0, \left|\overline{\mathbf{H}}_4\right| < 0$, etc. (i.e. the leading principal minors are all negative).

Now

1. If $L_\lambda = L_1 = L_2 = \ldots = L_n = 0$ and $\overline{\mathbf{H}}$ is negative definite, then the function $f(x_1, x_2, \ldots, x_n)$ has a local constrained maximum at $(x_1^*, x_2^*, \ldots, x_n^*)$.
2. If $L_\lambda = L_1 = L_2 = \ldots = L_n = 0$ and $\overline{\mathbf{H}}$ is positive definite, then the function $f(x_1, x_2, \ldots, x_n)$ has a local constrained minimum at $(x_1^*, x_2^*, \ldots, x_n^*)$.

**Example 23** *Consider the previous example*

$$\max_{x,y}\ 4xy - 2x^2 + y^2 \text{ subject to } 3x + y = 5$$

*The Lagrangian*

$$L(x, y, \lambda) = 4xy - 2x^2 + y^2 + \lambda\,(5 - 3x - y)$$

*We form the bordered Hessian:*

$$\begin{aligned}\overline{\mathbf{H}} &= \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 3 & 1 \\ 3 & -4 & 4 \\ 1 & 4 & 2 \end{bmatrix}\end{aligned}$$

*Check the SOC:*

$$\left|\overline{\mathbf{H}}_2\right| = \left|\overline{\mathbf{H}}\right| = 10 > 0$$

*Thus the bordered Hessian is negative definite and our optimum represents a constrained maximum.*

**Example 24** *The problem is to*

$$\max_{x_1,x_2} U = x_1x_2 + 2x_1 \text{ subject to } 4x_1 + 2x_2 = 60$$

*Set up the Lagrangian*

$$L(x_1, x_2, \lambda) = x_1x_2 + 2x_1 + \lambda(60 - 4x_1 - 2x_2)$$

*FOCs:*

$$\frac{\partial L}{\partial x_1} = x_2 + 2 - 4\lambda = 0 \quad (1)$$

$$\frac{\partial L}{\partial x_2} = x_1 - 2\lambda = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = 60 - 4x_1 - 2x_2 = 0 \quad (3)$$

*Solve simultaneously:*

$$\begin{aligned} \textit{From } (1) &: \quad x_2 = 4\lambda - 2 \\ \textit{From } (2) &: \quad x_1 = 2\lambda \end{aligned}$$

*Substitute these into* (12)*:*

$$\begin{aligned} 60 - 4x_1 - 2x_2 &= 0 \\ \Rightarrow 60 - 4(2\lambda) - 2(4\lambda - 2) &= 0 \\ 16\lambda &= 64 \\ \therefore \lambda^* &= 4 \end{aligned}$$

*If* $\lambda^* = 4, x_1^* = 8, x_2^* = 14, U^* = 128.$
*SOC:*

$$\begin{aligned} \overline{\mathbf{H}} &= \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -4 & -2 \\ -4 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \end{aligned}$$

$$\left|\overline{\mathbf{H}}_2\right| = \left|\overline{\mathbf{H}}\right| = 16 > 0$$

*Thus the bordered Hessian is negative definite and* $U^* = 128$ *represents a constrained maximum.*

**Example 25** *Find the extremum of $z = xy$ subject to $x + y = 6$.*

*Set up the Lagrangian*

$$L = xy + \lambda(6 - x - y)$$

*FOCs:*

$$\begin{aligned} \frac{\partial L}{\partial x} &= y - \lambda = 0 && (4) \\ \frac{\partial L}{\partial y} &= x - \lambda = 0 && (5) \\ \frac{\partial L}{\partial \lambda} &= 6 - x - y = 0 && (6) \end{aligned}$$

*Solve simultaneously. From (4) and (5):*

$$\lambda = x = y$$

*Substitute $y = x$ into (6):*

$$\begin{aligned} 6 - 2x &= 0 \\ \Rightarrow x^* &= 3 \end{aligned}$$

*Thus our constrained optimum is given by $x^* = 3, y^* = 3, \lambda^* = 3$ and $z^* = 9$.*

*SOC:*

$$\begin{aligned} \overline{\mathbf{H}} &= \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \end{aligned}$$

$$\left|\overline{\mathbf{H}}_2\right| = \left|\overline{\mathbf{H}}\right| = 2 > 0$$

*Thus the bordered Hessian is negative definite and $z^* = 9$ represents a constrained maximum.*

**Example 26** *Find the extremum of $z = x^2 + y^2$ subject to $x + 4y = 2$.*

*Set up the Lagrangian*

$$L = x^2 + y^2 + \lambda(2 - x - 4y)$$

*FOCs:*

$$\begin{aligned} \frac{\partial L}{\partial x} &= 2x - \lambda = 0 && (7) \\ \frac{\partial L}{\partial y} &= 2y - 4\lambda = 0 && (8) \\ \frac{\partial L}{\partial \lambda} &= 2 - x - 4y = 0 && (9) \end{aligned}$$

*Solve simultaneously.*

$$\textit{From } (7)\textit{:} \qquad x = \frac{1}{2}\lambda$$
$$\textit{From } (8)\textit{:} \qquad y = 2\lambda$$

*Substitute these into* (9)*:*

$$\begin{aligned} 2 - \frac{1}{2}\lambda - 4\,(2\lambda) &= 0 \\ \Rightarrow 4 - 17\lambda &= 0 \\ \therefore \lambda^* &= \frac{4}{17} \end{aligned}$$

*If* $\lambda^* = \frac{4}{17}, x^* = \frac{2}{17}, y^* = \frac{8}{17},$ *and* $z^* = \frac{4}{17}$.
*SOC:*

$$\begin{aligned} \overline{\mathbf{H}} &= \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & -4 \\ -1 & 2 & 0 \\ -4 & 0 & 2 \end{bmatrix} \end{aligned}$$

$$\left|\overline{\mathbf{H}}_2\right| = \left|\overline{\mathbf{H}}\right| = -34 < 0$$

*Thus the bordered Hessian is positive definite and* $z^* = \frac{4}{17}$ *represents a constrained minimum.*

## 6.3 An interpretation of the Lagrangian multiplier

The Lagrangian multiplier $\lambda^*$ gives a measure of the effect of a change in the constraint on the optimal value of the objective function.

## 6.4 Economic Applications

### 6.4.1 The utility maximising consumer

The consumer's problem is to maximise utility subject to a budget constraint:

$$\max_{x,y} U = U(x_1, x_2) \text{ subject to } x_1 p_1 + x_2 p_2 = m$$

The Lagrangian is:

$$L(x_1, x_2, \lambda) = U(x_1, x_2) + \lambda\,(m - x_1 p_1 - x_2 p_2)$$

FOCs:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= U_1 - \lambda p_1 = 0 && (10)\\
\frac{\partial L}{\partial y} &= U_2 - \lambda p_2 = 0 && (11)\\
\frac{\partial L}{\partial \lambda} &= m - x_1 p_1 - x_2 p_2 = 0 && (12)
\end{aligned}$$

$(10) \div (11)$ :

$$\begin{aligned}
\frac{U_1}{U_2} &= \frac{p_1}{p_2}\\
\Rightarrow \underbrace{-\frac{U_1}{U_2}}_{\substack{\text{slope of}\\ \text{indifference curve}}} &= \underbrace{-\frac{p_1}{p_2}}_{\substack{\text{slope of}\\ \text{budget constraint}}}
\end{aligned}$$

Recall that $\lambda^*$ measures the comparative static effect of the constraint constant on the optimal value of the objective function.

Hence, here

$$\lambda^* = \frac{\partial U^*}{\partial m}$$

Thus, $\lambda^*$ gives the marginal utility of money (budget money) when the consumer's utility is maximised.

SOC:

$$\overline{\mathbf{H}} = \begin{bmatrix} 0 & p_1 & p_2 \\ p_1 & U_{11} & U_{12} \\ p_2 & U_{21} & U_{22} \end{bmatrix}$$

For a constrained maximum

$$\left|\overline{\mathbf{H}}_2\right| = \left|\overline{\mathbf{H}}\right| = 2p_1 p_2 U_{12} - p_2^2 U_{11} - p_1^2 U_{22} > 0$$

### 6.4.2 Least cost combination of inputs

The firm has some choice of what technique to use when producing a given output level - e.g. a choice between a capital-intensive technique and a labour-intensive technique. The firm's cost function is then defined as the *minimal cost* of producing $Q$ units of output when input prices are $w$ and $r$.

Let the firm have the production function $F(K, L)$. The firm wants to choose an input combination $(K, L)$ so as to produce a given output $Q$ at minimal cost. Its problem is therefore to

$$\min_{K,L} rK + wL \text{ subject to } Q = F(K, L)$$

The Lagrangian is:

$$L(K, L, \lambda) = rK + wL + \lambda(Q - F(K, L))$$

FOCs:

$$\frac{\partial L}{\partial K} = r - \lambda F_K = 0 \tag{13}$$

$$\frac{\partial L}{\partial L} = w - \lambda F_L = 0 \tag{14}$$

$$\frac{\partial L}{\partial \lambda} = Q - F(K, L) = 0 \tag{15}$$

$(13) \div (14)$ :

$$\frac{r}{w} = \frac{F_K}{F_L}$$

$$\Rightarrow \underbrace{-\frac{r}{w}}_{\substack{\text{slope of} \\ \text{isocost}}} = \underbrace{-\frac{F_K}{F_L}}_{\substack{\text{slope of} \\ \text{isoquant (MRTS)}}}$$

Here $\lambda^*$ gives the marginal cost of production in the optimal state.

SOC:

$$\overline{\mathbf{H}} = \begin{bmatrix} 0 & F_K & F_L \\ F_K & -\lambda F_{KK} & -\lambda F_{KL} \\ F_L & -\lambda F_{LK} & -\lambda F_{LL} \end{bmatrix}$$

For a constrained minimum

$$\left|\overline{\mathbf{H}}_2\right| = \left|\overline{\mathbf{H}}\right| = \lambda\left(F_{KK}F_L^2 - 2F_{KL}F_KF_L + F_{LL}F_K^2\right) < 0$$

# 7 Inequality Constraints

The optimisation problems we have considered thus far all contain constraints that hold with equality.

However, often optimisation problems have constraints that take the form of inequalities rather than equalities. For example, the constraint may require that the consumption of a good is non-negative, i.e. $c \geq 0$.

We therefore need a new approach to solving these problems. A modification of the Lagrange multiplier model presents a technique for solving optimisation problems in which the constraints take the form of inequalities.

## 7.1 Non-negativity constraints

To begin with, consider a problem with nonnegativity restrictions on the choice variables, but with no other constraints. Taking the single-variable case, in particular, we have

$$\begin{aligned} \text{Maximise } y &= f(x) \\ \text{subject to } x &\geq 0 \end{aligned} \tag{16}$$

where $f$ is assumed to be differentiable.

Three situations may arise, as seen in Chiang Figure 13.1, page 403, 4th Edition.:

(a) A local maximum of $y$ occurs in the interior of the shaded feasible area, such as point A in Figure 13.1, then we have an *interior solution*. The FOC in this case is $\frac{dy}{dx} = f'(x) = 0$, same as usual.

(b) A local maximum can also occur on the vertical axis, shown in point B, where $x = 0$. Even here, where we have a *boundary solution*, the FOC $\frac{dy}{dx} = f'(x) = 0$ is still valid.

(c) A local maximum may take the position of points C or D, because to qualify as a local maximum the point simply has to be higher than the neighbouring points within the feasible region. Note that here $f'(x) < 0$. We can rule out $f'(x) > 0$ because if the curve is upward sloping, we can never have a maximum, even if that point is on the vertical axis, such as point E.

We can summarise these possibilities. In order for a value of $x$ to give a local maximum of $f$ in problem (16), it must satisfy one of the following three conditions:

$$f'(x) = 0 \quad \text{and} \quad x > 0 \quad \text{[point A]} \tag{17}$$

$$f'(x) = 0 \quad \text{and} \quad x = 0 \quad \text{[point B]} \tag{18}$$

$$f'(x) < 0 \quad \text{and} \quad x = 0 \quad \text{[points C and D]} \tag{19}$$

We can, in fact, consolidate these three conditions into a single statement:

$$f'(x) \leq 0 \qquad x \geq 0 \qquad \text{and} \qquad xf'(x) = 0 \tag{20}$$

The first inequality summarises the information about $f'(x)$ contained in (17) to (19) and so does the second inequality (it in fact merely restates the nonnegativity constraint). The third equation also summarises an important feature common to (17) to (19) and states that at least one of $x$ and $f'(x)$ must be zero, so that the product of the two must be zero. This is referred to as the *complementary slackness* between $x$ and $f'(x)$.

We can generalise this point to a problem containing $n$ choice variables:

$$\begin{aligned} \text{Maximise } y &= f(x_1, x_{2,\ldots,}x_n) \\ \text{subject to } x_j &\geq 0 \end{aligned}$$

The classical FOC $f_1 = f_2 = \ldots = f_n$ must be similarly modified. The required FOC is now:

$$f_j \leq 0 \qquad x_j \geq 0 \qquad \text{and} \qquad x_j f_j = 0 \qquad (j = 1, 2, \ldots, n)$$

where $f_j$ is the partial derivative $\dfrac{\partial y}{dx_j}$.

### 7.2 General inequality constraints

Let $f(x,y)$ and $g(x,y)$ be functions of two variables. We wish to

$$\begin{array}{ll} \text{Maximise} & y = f(x,y) \\ \text{subject to} & g(x,y) \leq 0 \end{array} \tag{21}$$

Note that $g(x,y) \leq 0$ may be regarded as the general form for a weak inequality involving the variables $x, y$. For example, the inequality

$$F(x,y) + G(x) \geq H(y) + 7$$

can be rewritten in that form by

$$g(x,y) = 7 - F(x,y) - G(x) + H(y) \leq 0$$

Suppose that the constrained maximum for problem (21) is obtained when $x = x^*$ and $y = y^*$. Then we have two cases to consider:

1. **Case I** $\quad g(x^*, y^*) < 0$

   In this case the constraint is said to be *slack*, or *inactive*, at $(x^*, y^*)$. Assuming as usual that $g$ is continuous, $g(x,y) < 0$ for all points $(x,y)$ sufficiently close to $(x^*, y^*)$ : but then $f(x,y) \leq f(x^*, y^*)$ for all such $(x,y)$. Hence $f$ has a local unconstrained maximum and therefore a critical point at $(x^*, y^*)$.

2. **Case II** $\quad g(x^*, y^*) = 0$

   In this case the constraint is said to be *tight*, or *active*, at $(x^*, y^*)$. In particular, $(x^*, y^*)$ is the point which maximises $f(x,y)$ subject to $g(x,y) = 0$ : hence there exists a multiplier $\lambda$ such that the Lagrangian $f - \lambda g$ has a critical point at $(x^*, y^*)$.

   Now recall that $(x^*, y^*)$ maximises $f(x,y)$ subject to the inequality constraint $g(x,y) \leq 0$, so that the feasible set is much larger than it would be if we had imposed the constraint $g(x,y) = 0$ at the outset. This provides us with an additional piece of information:

   if we let $v(b)$ denote the maximal value of $f(x,y)$ subject to $g(x,y) = b$,

   then $f(x^*, y^*) \geq v(b)$ whenever $b < 0$.

   But $f(x^*, y^*) = v(0)$, so $v(0) \geq v(b)$ whenever $b < 0$;

   it follows that $v'(0) \geq 0$.

   Now we know that $v'(0)$ is the Lagrange multiplier $\lambda$,

   so $\lambda \geq 0$.

   So, in case II the Lagrange method holds, with the additional information that the multiplier is non-negative.

We can summarise what happens at the constrained maximum $(x^*, y^*)$ as follows:

- In case II, $g(x^*, y^*) = 0$, and there exists a Lagrange multiplier $\lambda$ such that

$$\begin{aligned} \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} &= 0, \\ \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} &= 0 \\ \text{and } \lambda &\geq 0 \end{aligned}$$

- In case I, $g(x^*, y^*) < 0$, and $f$ has an unconstrained local maximum at $(x^*, y^*)$. Therefore, at that point,

$$\begin{aligned} \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} &= 0, \\ \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} &= 0 \\ \text{where } \lambda &= 0 \end{aligned}$$

These results can be combined as follows:

**Proposition 1** *Let the Lagrangian for problem* (21) *be defined as*

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

*and let* $x = x^*, y = y^*$ *be a solution of the problem. Then there exists a number* $\lambda^*$ *with the following properties:*

1. $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$ *at* $(x^*, y^*, \lambda^*)$;
2. $\lambda^* \geq 0, g(x^*, y^*) \leq 0$ *and at least one of these two quantities is zero.*

Note that condition (2) states, among other things, that at least one of the two numbers $\lambda^*$ and $g(x^*, y^*)$ is zero: in short,

$$\lambda^* g(x^*, y^*) = 0$$

This property is known as *complementary slackness.*

## 7.3 The Kuhn-Tucker theorem

The necessary conditions (1) and (2) for a constrained maximum may be generalised to the case of many variables and constraints.

Suppose that $f, g_1, \ldots, g_m$ are functions of $n$ variables: our problem is to

$$\begin{array}{lll} \text{Maximise} & f(\mathbf{x}) & \\ \text{subject to} & g_i(\mathbf{x}) \leq 0 & (i = 1, \ldots, m) \end{array}$$

We define the Lagrangian for this problem to be

$$(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) - \lambda_1 g_1(\mathbf{x}) - \ldots - \lambda_m g_m(\mathbf{x})$$

Suppose the maximum value of $f(\mathbf{x})$ subject to the constraints is obtained when $\mathbf{x} = \mathbf{x}^*$. There exists multipliers $\lambda_1^*, \ldots, \lambda_m^*$ with the following properties:

1. At $(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*)$, $\dfrac{\partial}{\partial x_j} = 0$ for $j = 1, \ldots, n$
2. For $i = 1, \ldots, m, \lambda_i^* \geq 0, g_i(\mathbf{x}^*) \leq 0$ and $\lambda_i^* g_i(\mathbf{x}^*) = 0$.

These results are known as the *Kuhn-Tucker theorem.* Conditions (1) and (2) are known as the *Kuhn-Tucker conditions* for the constrained maximisation problem.

### 7.3.1 Points about the Kuhn-Tucker theorem

1. *Mixed constraints*:

   The theorem can be extended to the case where equations as well as inequalities appear among the constraints. The necessary conditions for a constrained maximum then emerge as mixture of those we have already discussed.

   Suppose, for example, that we wish to maximise $f(\mathbf{x})$ subject to the constraints $g(\mathbf{x}) = 0$ and $h(\mathbf{x}) \leq 0$. Then there exist multipliers $\lambda$ and $\mu$ such that at the optimal $\mathbf{x}$ :

   (a) $\dfrac{\partial}{\partial x_j} [f(\mathbf{x}) - \lambda g(\mathbf{x}) - \mu h(\mathbf{x})] = 0$ for $j = 1, \ldots, n$;

   (b) $g(\mathbf{x}) = 0$

   (c) $\mu \geq 0, h(\mathbf{x}) \leq 0$ and at least one of them is zero.

   Notice that the condition (c) refers only to $\mu$, the multiplier associated with the inequality constraint; the other multiplier $\lambda$ may be positive, negative or zero.

2. *Non-negativity constraints*:

   The Kuhn-Tucker theorem can also be extended to the case where some or all of the components of $\mathbf{x}$ are required to be non-negative. The extension consists of a modification of the first Kuhn-Tucker condition similar to that in (20).

   For example, if $x_1$ is required to be non-negative, then the condition $\frac{\partial}{\partial x_1} = 0$ is replaced by $\frac{\partial}{\partial x_1} \leq 0$, with equality if $x_1 > 0$.

   Or you can just proceed as normal, using $x_1 \geq 0 \Rightarrow -x_1 \leq 0$

3. *Minimisation:*

   Since minimising $f(\mathbf{x})$ is equivalent to maximising $-f(\mathbf{x})$, we write the Lagrangian as above, replacing $f$ with $-f$.

4. *So what?*

   The theorem is not in itself a great help in actually finding constrained maxima. However, under appropriate assumptions about convexity, the Kuhn-Tucker conditions are sufficient as well as necessary for a maximum; and if these assumptions are satisfied, the conditions may be used to find the maximum. We end with a brief discussion of this point.

## 7.4 Sufficient conditions

The main theorem on sufficient conditions for a maximum with inequality constraints is as follows:

**Theorem 1** *Suppose we wish to maximise the function $f(\mathbf{x})$ subject to the constraints $g_i(\mathbf{x}) \leq 0$ for $i = 1, \ldots, m$. Suppose the Kuhn-Tucker conditions are satisfied at the point $\mathbf{x}^*$. Then the constrained maximum is attained at $\mathbf{x}^*$ if the following conditions are also satisfied:*

1. *$f$ is a quasi-concave function, and* either *$f$ is a monotonic transformation of a concave function* or *$Df(\mathbf{x}^*) \neq 0$, or both.*

2. *the functions $-g_1, \ldots, -g_m$ are all quasi-concave (or equivalently, that the functions $g_1, \ldots, g_m$ are all quasi-convex).*

We note that linear functions are quasi-concave, and concave functions satisfy condition (1). Thus, *the Kuhn-Tucker conditions are sufficient for an optimum when a concave function is maximised subject to linear constraints.*

**Example 27** *Maximise*

$$-(x_1-4)^2-(x_2-4)^2$$

*subject to*

$$x_1+x_2\le 4,\qquad x_1+3x_2\le 9$$

*Rewrite the constraints so they are in the form* $g_i(\mathbf{x})\le 0$ *:*

$$x_1+x_2-4\le 0,\qquad x_1+3x_2-9\le 0$$

*Set up the Lagrangian*

$$L=-(x_1-4)^2-(x_2-4)^2-\lambda_1(x_1+x_2-4)-\lambda_2(x_1+3x_2-9)$$

*The Kuhn-Tucker conditions are:*

$$\frac{\partial L}{\partial x_1} = -2(x_1-4)-\lambda_1-\lambda_2=0 \qquad (22)$$

$$\frac{\partial L}{\partial x_2} = -2(x_2-4)-\lambda_1-3\lambda_2=0 \qquad (23)$$

$$\lambda_1 \ge 0, x_1+x_2\le 4 \text{ and } \lambda_1(x_1+x_2-4)=0$$

$$\lambda_2 \ge 0, x_1+3x_2\le 9 \text{ and } \lambda_2(x_1+3x_2-9)=0$$

*What are the solutions to these conditions?*

*Start by looking at the two conditions* $\lambda_1(x_1+x_2-4)=0$ *and* $\lambda_2(x_1+3x_2-9)=0$*. These two conditions yield the following* four *cases:*

***Case 1****:* $\lambda_1>0,\lambda_2>0$

*Then* $x_1+x_2-4=0$ *and* $x_1+3x_2-9=0$*,*

*which gives* $x_1=\frac{3}{2}, x_2=\frac{5}{2}$*.*

*Then equations* (22) *and* (23) *are:*

$$\begin{aligned}5-\lambda_1-\lambda_2 &= 0\\ 3-\lambda_1-3\lambda_2 &= 0\end{aligned}$$

*which implies that* $\lambda_1=6$ *and* $\lambda_2=-1$*, which violates the condition that* $\lambda_2>0$*.*

***Case 2****:* $\lambda_1>0,\lambda_2=0$

*Then* $x_1+x_2-4=0$*.*

*Substituting* $\lambda_2=0$ *into equations* (22) *and* (23) *gives:*

$$\begin{aligned}-2(x_1-4)-\lambda_1 &= 0\\ -2(x_2-4)-\lambda_1 &= 0\end{aligned}$$

*which together with* $x_1+x_2-4=0$ *gives* $x_1=x_2=2$ *and* $\lambda_1=4$*. All the conditions are satisfied so* $x_1=x_2=2$ *with* $\lambda_1=4,\lambda_2=0$ *is a possible solution.*

***Case 3:*** *$\lambda_1 = 0, \lambda_2 > 0$*
*Then $x_1 + 3x_2 - 9 = 0$*
*Substituting $\lambda_1 = 0$ into equations (22) and (23) gives:*

$$\begin{aligned} -2(x_1 - 4) - \lambda_2 &= 0 \\ -2(x_2 - 4) - 3\lambda_2 &= 0 \end{aligned}$$

*which together with $x_1 + 3x_2 - 9 = 0$ gives $x_1 = \dfrac{33}{10}, x_2 = \dfrac{19}{10}$, which violates $x_1 + x_2 \leq 4$.*
***Case 4:*** *$\lambda_1 = 0, \lambda_2 = 0$*
*Substituting $\lambda_1 = 0, \lambda_2 = 0$ into equations (22) and (23) gives:*

$$\begin{aligned} -2(x_1 - 4) &= 0 \\ -2(x_2 - 4) &= 0 \end{aligned}$$

*which is satisfied if and only if $x_1 = x_2 = 4$, which violates $x_1 + x_2 \leq 4$.*

*So $x_1 = x_2 = 2$ is the single solution of the problem and the value of the objective function at this point is $-8$.*

**Example 28** *Maximise*

$$-x_1^2 - x_1x_2 - x_2^2$$

*subject to*

$$x_1 - 2x_2 \leq -1, \qquad 2x_1 + x_2 \leq 2$$

*Rewrite the constraints so they are in the form $g_i(\mathbf{x}) \leq 0$ :*

$$x_1 - 2x_2 + 1 \leq 0, \quad 2x_1 + x_2 - 2 \leq 0$$

*Set up the Lagrangian*

$$L = -x_1^2 - x_1x_2 - x_2^2 - \lambda_1(x_1 - 2x_2 + 1) - \lambda_2(2x_1 + x_2 - 2)$$

*The Kuhn-Tucker conditions are:*

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= -2x_1 - x_2 - \lambda_1 - 2\lambda_2 = 0 && (24) \\ \frac{\partial L}{\partial x_2} &= -x_1 - 2x_2 + 2\lambda_1 - \lambda_2 = 0 && (25) \\ \lambda_1 &\geq 0, x_1 - 2x_2 \leq -1 \text{ and } \lambda_1(x_1 - 2x_2 + 1) = 0 \\ \lambda_2 &\geq 0, 2x_1 + x_2 \leq 2 \text{ and } \lambda_2(2x_1 + x_2 - 2) = 0 \end{aligned}$$

*Consider the possible cases in turn:*

***Case 1:*** $\lambda_1 > 0, \lambda_2 > 0$

*Then* $x_1 - 2x_2 + 1 = 0$ *and* $2x_1 + x_2 - 2 = 0$,

*which gives* $x_1 = \frac{3}{5}, x_2 = \frac{4}{5}$.

*Then equations* (24) *and* (25) *are:*

$$\begin{aligned} -2 - \lambda_1 - 2\lambda_2 &= 0 \\ -\frac{11}{5} + 2\lambda_1 - \lambda_2 &= 0 \end{aligned}$$

*which implies that* $\lambda_1 = \frac{12}{25}$ *and* $\lambda_2 = -\frac{31}{25}$, *which violates the condition that* $\lambda_2 > 0$.

***Case 2:*** $\lambda_1 > 0, \lambda_2 = 0$

*Then* $x_1 - 2x_2 + 1 = 0$.

*Substituting* $\lambda_2 = 0$ *into equations* (24) *and* (25) *gives:*

$$\begin{aligned} -2x_1 - x_2 - \lambda_1 &= 0 \\ -x_1 - 2x_2 + 2\lambda_1 &= 0 \end{aligned}$$

*which together with* $x_1 - 2x_2 + 1 = 0$ *gives* $x_1 = -\frac{4}{14}, x_2 = \frac{5}{14}$ *and* $\lambda_1 = \frac{3}{14}$. *All the conditions are satisfied so* $x_1 = -\frac{4}{14}, x_2 = \frac{5}{14}$ *with* $\lambda_1 = \frac{3}{14}, \lambda_2 = 0$ *is a possible solution.*

***Case 3:*** $\lambda_1 = 0, \lambda_2 > 0$

*Then* $2x_1 + x_2 - 2 = 0$.

*Substituting* $\lambda_1 = 0$ *into equations* (24) *and* (25) *gives:*

$$\begin{aligned} -2x_1 - x_2 - 2\lambda_2 &= 0 \\ -x_1 - 2x_2 - \lambda_2 &= 0 \end{aligned}$$

*which together with* $2x_1 + x_2 - 2 = 0$ *gives* $x_1 = 1, x_2 = 0$ *and* $\lambda_2 = -1$. *Since* $\lambda_2 < 0$ *this case does not satisfy the Kuhn-Tucker constraints.*

***Case 4:*** $\lambda_1 = 0, \lambda_2 = 0$

*Substituting* $\lambda_1 = 0, \lambda_2 = 0$ *into equations* (24) *and* (25) *gives:*

$$\begin{aligned} -2x_1 - x_2 &= 0 \\ -x_1 - 2x_2 &= 0 \end{aligned}$$

*which is satisfied if and only if* $x_1 = x_2 = 0$, *which violates* $x_1 - 2x_2 \leq -1$.

*The unique solution to the problem is therefore* $x_1 = -\frac{4}{14}, x_2 = \frac{5}{14}$ *with* $\lambda_1 = \frac{3}{14}$, $\lambda_2 = 0$.

**Example 29** *Maximise the objective function*

$$f(x,y) = x - \frac{x^2}{2} + y^2$$

*subject to the constraints*

$$\frac{x^2}{2} + y^2 \leq \frac{9}{8}, \qquad -y \leq 0$$

*Notice that the constraint that $y$ is nonnegative (i.e. $y \geq 0$) has been written in a way to make it conformable with the set-up of the problem given above.*

*We can rewrite the other constraint in the form $g_i(\mathbf{x}) \leq 0$*

$$\frac{x^2}{2} + y^2 - \frac{9}{8} \leq 0, \qquad -y \leq 0$$

*Set up the Lagrangian*

$$L = x - \frac{x^2}{2} + y^2 - \lambda\left(\frac{x^2}{2} + y^2 - \frac{9}{8}\right) - \mu(-y)$$

*The Kuhn-Tucker conditions are:*

$$\frac{\partial L}{\partial x} = 1 - x - \lambda x = 0 \tag{26}$$

$$\frac{\partial L}{\partial y} = 2y - 2\lambda y + \mu = 0 \tag{27}$$

$$\lambda \geq 0, \frac{x^2}{2} + y^2 \leq \frac{9}{8} \text{ and } \lambda\left(\frac{x^2}{2} + y^2 - \frac{9}{8}\right) = 0$$

$$\mu \geq 0, -y \leq 0 \text{ and } \mu(-y) = 0$$

*We proceed by considering the solutions that arise in the four cases corresponding to the four possible ways in which the complementary slackness conditions may be met. These are:*

***Case 1:*** $\lambda > 0, \mu > 0$
***Case 2:*** $\lambda > 0, \mu = 0$
***Case 3:*** $\lambda = 0, \mu > 0$
***Case 4:*** $\lambda = 0, \mu = 0$

*We will then compare the value of the objective function for each solution to find the optimal choice of $x$ and $y$.*

***Case 1:*** *$\lambda > 0, \mu > 0$*
*Then* $\frac{x^2}{2} + y^2 - \frac{9}{8} = 0$ *and* $-y = 0$.
*If* $y = 0$, *then equation (27) becomes*

$$\begin{aligned} 2y - 2\lambda y + \mu &= 0 \\ \Rightarrow \mu &= 0 \end{aligned}$$

*which violates our condition that* $\mu > 0$ *and there is no solution for which* $\lambda > 0, \mu > 0$.
***Case 2:*** $\lambda > 0, \mu = 0$
*Then* $\frac{x^2}{2} + y^2 - \frac{9}{8} = 0$.
*Substituting* $\mu = 0$ *into equation (27) gives:*

$$\begin{aligned} 2y - 2\lambda y &= 0 \\ \Rightarrow 2y(1 - \lambda) &= 0 \\ \Rightarrow \lambda &= 1 \text{ or } y = 0 \end{aligned}$$

*We have two subcases so we consider them one at a time:*

- ***If*** $\lambda = 1$, *then equation (26) gives*

$$\begin{aligned} 1 - x - \lambda x &= 0 \\ \Rightarrow 1 - x - x &= 0 \\ \Rightarrow x &= \frac{1}{2} \end{aligned}$$

*If* $\lambda = 1$ *and* $x = \frac{1}{2}$, *complementary slackness requires*

$$\begin{aligned} \frac{x^2}{2} + y^2 - \frac{9}{8} &= 0 \\ 1\left(\frac{\left(\frac{1}{2}\right)^2}{2} + y^2 - \frac{9}{8}\right) &= 0 \\ \Rightarrow y^2 - 1 &= 0 \\ \Rightarrow y &= \pm 1 \\ \text{But } -y &< 0 \Rightarrow y = 1 \text{ not } -1 \end{aligned}$$

*All the conditions are satisfied and* $x = \frac{1}{2}, y = 1$ *is a possible solution and the value of the objective function is* $f\left(\frac{1}{2}, 1\right) = \frac{11}{8}$

- ***If*** $y = 0$, *then*

$$\begin{aligned}\frac{x^2}{2} + y^2 - \frac{9}{8} &= 0\\ \Rightarrow \frac{x^2}{2} &= \frac{9}{8}\\ \Rightarrow x &= \pm\sqrt{\frac{9}{4}}\\ x &= \pm\frac{3}{2}\end{aligned}$$

  - *If* $x = \frac{3}{2}$, *then equation* (26) *gives*

$$\begin{aligned}1 - \frac{3}{2} - \lambda\frac{3}{2} &= 0\\ \Rightarrow & \lambda = -\frac{1}{3}\end{aligned}$$

    *which violates* $\lambda > 0$.
  - *If* $x = -\frac{3}{2}$, *then equation* (26) *gives*

$$\begin{aligned}1 + \frac{3}{2} + \lambda\frac{3}{2} &= 0\\ \Rightarrow & \lambda = -\frac{5}{3}\end{aligned}$$

    *which violates* $\lambda > 0$.

  *There is no solution here where* $y = 0$.

***Case 3:*** $\lambda = 0, \mu > 0$
*Then* $-y = 0$.
*Substituting* $y = 0$ *into equation* (27) *gives:*

$$\begin{aligned}2y - 2\lambda y + \mu &= 0\\ \Rightarrow \mu &= 0\end{aligned}$$

*But this violates the condition that* $\mu > 0$.
***Case 4:*** $\lambda = 0, \mu = 0$
*Substituting* $\lambda = 0, \mu = 0$ *into equations* (26) *and* (27) *gives:*

$$\begin{aligned}1 - x &= 0\\ y &= 0\end{aligned}$$

*which is satisfied if and only if* $x = 1, y = 0$.

*When* $x = 1, y = 0$*, both the constraints* $\frac{x^2}{2} + y^2 \leq \frac{9}{8}$ *and* $-y \leq 0$ *are satisfied.*

*The value of the objective function is* $f(1,0) = \frac{1}{2}$

*The last step is to rank the solutions by the value of the objective function. We have two possible solutions:*

*Case 2:* $x = \frac{1}{2}, y = 1$ *and* $f\left(\frac{1}{2}, 1\right) = \frac{11}{8}$.

*Case 4:* $x = 1, y = 0$ *and* $f(1,0) = \frac{1}{2}$

*So the optimum is* $x = \frac{1}{2}, y = 1$ *since* $f\left(\frac{1}{2}, 1\right) > f(1,0)$*. At the optimal solution* $\frac{x^2}{2} + y^2 = \frac{9}{8}$ *and this constraint is binding, whereas* $-y < 0$ *and this constraint is not binding.*

**Example 30**

$$2\sqrt{x} + 2\sqrt{y}$$

*subject to*

$$2x + y \leq 3, \qquad x + 2y \leq 3$$

*Rewrite the constraints so they are in the form* $g_i(\mathbf{x}) \leq 0$ *:*

$$2x + y - 3 \leq 0, \qquad x + 2y - 3 \leq 0$$

*Set up the Lagrangian*

$$L = 2\sqrt{x} + 2\sqrt{y} - \lambda(2x + y - 3) - \mu(x + 2y - 3)$$

*The Kuhn-Tucker conditions are:*

$$\frac{\partial L}{\partial x} = x^{-1/2} - 2\lambda - \mu = 0 \tag{28}$$

$$\frac{\partial L}{\partial y} = y^{-1/2} - \lambda - 2\mu = 0 \tag{29}$$

$$\lambda \geq 0, 2x + y \leq 3 \text{ and } \lambda(2x + y - 3) = 0$$

$$\mu \geq 0, x + 2y \leq 3 \text{ and } \mu(x + 2y - 3) = 0$$

*What are the solutions to these conditions?*

*Start by looking at the two conditions* $\lambda(2x + y - 3) = 0$ *and* $\mu(x + 2y - 3) = 0$*. These two conditions yield the following* four *cases:*

***Case 1***: $\lambda > 0, \mu > 0$

*Then* $2x + y - 3 = 0$ *and* $x + 2y - 3 = 0$,

*which gives* $x = y = 1$.

*Then equations* (28) *and* (29) *are:*

$$\begin{aligned} 1 - 2\lambda - \mu &= 0 \\ 1 - \lambda - 2\mu &= 0 \end{aligned}$$

*which happens if and only if* $\lambda = \mu = \frac{1}{3}$. *But then* $\lambda$ *and* $\mu$ *are positive numbers, so all conditions are met.*

***Case 2***: $\lambda > 0, \mu = 0$

*Then* $2x + y - 3 = 0$.

*Substituting* $\mu = 0$ *into equations* (28) *and* (29) *gives:*

$$\begin{aligned} x^{-1/2} - 2\lambda &= 0 \\ y^{-1/2} - \lambda &= 0 \end{aligned}$$

*which together with* $2x + y - 3 = 0$ *gives* $x = \frac{1}{2}, y = 2$.

*But then* $x + 2y - 3 = \frac{3}{2}$ *which violates the condition* $x + 2y \leq 3$.

***Case 3***: $\lambda = 0, \mu > 0$

*Then* $x + 2y - 3 = 0$.

*Substituting* $\lambda = 0$ *into equations* (28) *and* (29) *gives:*

$$\begin{aligned} x^{-1/2} - \mu &= 0 \\ y^{-1/2} - 2\mu &= 0 \end{aligned}$$

*which together with* $x + 2y - 3 = 0$ *gives* $x = 2, y = \frac{1}{2}$.

*But then* $2x + y = \frac{9}{2}$ *which violates the condition* $2x + y \leq 3$.

***Case 4***: $\lambda = 0, \mu = 0$

*Substituting* $\lambda = 0, \mu = 0$ *into equations* (28) *and* (29) *gives:*

$$\begin{aligned} x^{-1/2} &= 0 \\ y^{-1/2} &= 0 \end{aligned}$$

*which is satisfied if and only if* $x = y = 0$.

*The conditions* $2x + y \leq 3$ *and* $x + 2y \leq 3$ *are satisfied if* $x = y = 0$.

*So possible solutions are found under Cases 1 and 4.*

*Case 1:* $x = y = 1$ *and* $2\sqrt{x} + 2\sqrt{y} = 4$.

*Case 4:* $x = y = 0$ *and* $2\sqrt{x} + 2\sqrt{y} = 0$.

*The optimal solution is the one that yields the highest value of the objective function, i.e.* $x = y = 1$. *At this solution, both constraints are binding.*

## References

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