Mathematics for Economists

Comparative Statics

Section 2: Comparative Statics

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As we've said before, a key concept in economics is that of equilibrium. A large part of the mathematical modelling we do in this regard is concerned with comparative statics, that is, the comparison of different equilibrium states that are associated with different sets of values of parameters and exogenous variables. To make such a comparison, we always start by assuming an initial equilibrium state. We then allow some kind of disequilibrating change in the model (through some change in a parameter or an exogenous variable). When this occurs, the initial equilibrium will of course be upset, and so the endogenous variables will have to adjust. The question we concern ourselves with in comparative statics is "How will the new equilibrium position compare with the old?"

NB: When we study comparative statics, we simply compare the initial (pre-change) equilibrium position to the post-change equilibrium position. We cannot say anything about the process of adjustment.

Our comparative static analysis can be either *quantitative* or *qualitative*. If our analysis is purely qualitative, this means that we will only be able to talk about the direction of the change that occurs. If it is quantitative, we will actually be able to talk about the magnitude of the change that has occurred. (Obviously, if we know the magnitude, we will also know the direction of the change, so in effect, the quantitative analysis involves a qualitative element as well).

The crux of all of this is that in doing comparative statics, we are looking for a rate of change, namely the *rate of change* of the equilibrium value of an endogenous variable with respect to a change in the particular parameter or exogenous variable. (How does the endogenous variable change in response to a change in the exogenous variable, or a change in the parameter) To get at this, we will make use of the concept of a **derivative**, a concept that is concerned with rates of change.

1 The Derivative

1.1 The Difference Quotient

When x changes from the value x_0 to a new value x_1, the change is given by the difference $x_1 - x_0$. We use the symbol Δ to denote the change, hence we can write $\Delta x = x_1 - x_0$.

It is standard to use the notation $f(x_i)$ to represent the value of the function $f(x)$ when $x = x_i$. For example, if $f(x) = x^2 - 3$, then $f(1) = (1)^2 - 3 = -2$ and $f(2) = (2)^2 - 3 = 1$.

Consider the simple function $y = f(x)$.

The initial value of x is x_0, and so $y = f(x_0)$.

When x changes to a new value $(x_0 + \Delta x)$, the value of y at this point is $y = f(x_0 + \Delta x)$.

The *difference quotient* gives us the change in y per unit change in x:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

This gives us the average rate of change of y.

Example 1 *Consider the linear function $y = f(x) = 2x + 1$. Suppose x changes from x_0 to $(x_0 + \Delta x)$, then the difference quotient is given by*

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ &= \frac{[2(x_0 + \Delta x) + 1] - [2x_0 + 1]}{\Delta x} \\ &= \frac{2\Delta x}{\Delta x} \\ &= 2\end{aligned}$$

Example 2 *Consider the quadratic function $y = f(x) = 2x^2 - 1$. Suppose x changes from x_0 to $(x_0 + \Delta x)$, then the difference quotient is given by*

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ &= \frac{\left[2(x_0 + \Delta x)^2 - 1\right] - \left[2x_0^2 - 1\right]}{\Delta x} \\ &= \frac{2\left(x_0^2 + 2x_0\Delta x + (\Delta x)^2 - 1\right) - 2x_0^2 + 1}{\Delta x} \\ &= \frac{4x_0\Delta x + (\Delta x)^2}{\Delta x} \\ &= 4x_0 + \Delta x\end{aligned}$$

1.2 The Derivative

We are usually interested in the rate of change of y when Δx is very small. In this case, it is possible to obtain an approximation of the difference quotient by dropping all the terms involving Δx. So for example, for our function $y = f(x) = 2x^2 - 1$ above, we could approximate the difference quotient by taking $4x_0$ (we're effectively treating Δx as infinitesimally small). As Δx approaches 0, (i.e. it gets closer and closer to zero but never actually reaches it), the difference quotient $4x_0 + \Delta x$ will approach $4x_0$.

We can express this idea formally as follows:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (4x_0 + \Delta x) = 4x_0$$

and we read this as "the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero is $4x_0$".

If, as $\Delta x \to 0$, the limit of the difference quotient $\frac{\Delta y}{\Delta x}$ indeed exists, that limit is called the *derivative* of the function $y = f(x)$. The process of obtaining the derivative is known as *differentiation*.

Because the derivative is just the limit of the difference quotient (which measures a rate of change), the derivative is also a measure of a rate of change. However, because the change in x is so small ($\Delta x \to 0$), the derivative actually measures the *instantaneous rate of change*.

There are two common ways to denote a derivative. Given an original function $y = f(x)$, we can denote its derivative as follows: $f'(x)$ (or simply f') or $\frac{dy}{dx}$. Using these notations we may define the derivative of a function $y = f(x)$ as follows:

$$\frac{dy}{dx} \equiv f'(x) \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

2 The Derivative and the Slope of a Curve

One of the most common uses of the concept of a derivative in economics is to tell us something about the slope of a curve. For example, suppose we have a total cost function, where $C = f(Q)$. From economic theory, we know that the marginal cost (MC) is defined as the change in total cost resulting from a one-unit change in output (or quantity). In other words, $MC = \frac{\Delta C}{\Delta Q}$.

This should look familiar! We assume that ΔQ is very small, and thus, we can approximate $\frac{\Delta C}{\Delta Q}$ by taking its limit as $\Delta Q \to 0$, i.e.

$$\frac{dC}{dQ} \equiv f'(Q) \equiv \lim_{\Delta Q \to 0} \frac{\Delta C}{\Delta Q}$$

This is also measured by the slope of the total cost curve.

Note: *the slope of a curve is the geometric equivalent of the concept of a derivative.*

3 Limits and Continuity

The function $f(x)$ is differentiable if the limit of the difference quotient as $\Delta x \to 0$ exists, i.e. $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ exists. Not all functions are differentiable, and we now look at which functions can and cannot be differentiated.

First, let's look at the idea of limits more closely. The statement

$$f(x) \text{ tends to the limit } \ell \text{ as } x \text{ approaches } x_0$$

means that we may make $f(x)$ as close as we wish to ℓ for all x sufficiently close (but not equal) to x_0.

We write this as

$$f(x) \to \ell \text{ as } x \to x_0$$

and

$$\lim_{x \to x_0} f(x) = \ell$$

Importantly

$$\lim_{x \to x_0} f(x) \text{ **may or may not** be equal to } f(x_0)$$

Example 3 *Let $f(x) = 2x + 3$. Then $f(x)$ is close to 5 whenever x is close to 1. Hence*

$$\lim_{x \to 1} f(x) = 5 = f(1).$$

Example 4 *Let*

$$f(x) = \begin{cases} +1, & \text{when } x \neq 4 \\ -1, & \text{when } x = 4 \end{cases}$$

Since $f(x) = 1$ whenever $x \neq 4$, however close x is to 4, $f(x) \to 1$ as $x \to 4$. But $f(4) = -1$. Therefore

$$\lim_{x \to 4} f(x) \neq f(4).$$

Definition 1 *We say that the function $f(x)$ is **continuous** at x_0 if*

1. *The point x_0 is in the domain of f, i.e. $f(x_0)$ is defined.*
2. *The function has a limit as $x \to x_0$, i.e. $\lim_{x \to x_0} f(x)$ exists.*
3. *The limit as $x \to x_0$ must be equal in value to $f(x_0)$, i.e. $\lim_{x \to x_0} f(x) = f(x_0)$.*

So the function $f(x) = 2x + 3$ is continuous at $x = 1$, but the function $f(x) = \begin{cases} +1, & \text{when } x \neq 4 \\ -1, & \text{when } x = 4 \end{cases}$ is **discontinuous** (i.e. not continuous) at $x = 4$.

Definition 2 *We say that f is a **continuous function** if it is continuous at x for every x. Geometrically,* a function is continuous if its graph may be drawn without lifting the pencil from the paper.

4 Differentiability

We now return to the question of when differentiation is possible.

Definition 3 *A function is said to* differentiable *at a particular point if the derivative of the function can be found at that point.*

Definition 4 *A* differentiable function *is one that is differentiable at every point.*

Every differentiable function is continuous. On the other hand, **not all continuous functions are differentiable**. For example, the function $f(x) = |x|$ is a continuous function but is not differentiable at $x = 0$.

Continuity is a necessary but not sufficient condition for differentiability.

5 Rules of Differentiation

The good news is that we do not have to take the limit of the difference quotient each time we want calculate a derivative. We can use the rules of differentiation to help us.

5.1 Constant Function Rule

$$\begin{aligned} \text{If } y &= f(x) = c \text{ (where } c \text{ is a constant)} \\ \text{then } \frac{dy}{dx} &= f'(x) = 0 \end{aligned}$$

Example 5 *If $y = f(x) = 7$, then $\frac{dy}{dx} = f'(x) = 0$.*

5.2 Constant Factor Rule

$$\begin{aligned} \text{If } y &= cf(x) \\ \text{then } \frac{dy}{dx} &= cf'(x) \end{aligned}$$

5.3 Power Function Rule

$$\begin{aligned} \text{If } y &= f(x) = x^n \\ \text{then } \frac{dy}{dx} &= f'(x) = nx^{n-1} \end{aligned}$$

Example 6 *If* $y = f(x) = x^2$, *then* $\frac{dy}{dx} = f'(x) = 2x$.

Example 7 *If* $y = f(x) = 5x^2$, *then* $\frac{dy}{dx} = f'(x) = 5(2x) = 10x$.

Example 8 *If* $y = f(x) = 3x^3$, *then* $\frac{dy}{dx} = f'(x) = 3\left(3x^2\right) = 9x^2$.

5.4 Sum-Difference Rule

$$\begin{aligned} \text{If } y &= f(x) \pm g(x) \pm h(x) \\ \text{then } \frac{dy}{dx} &= f'(x) \pm g'(x) \pm h'(x) \end{aligned}$$

Example 9 *If* $y = f(x) = 3x^2 + 4x - 1$, *then* $\frac{dy}{dx} = f'(x) = 6x + 4$.

Example 10 *If* $y = f(x) = 6x^3 + 2x^2 + 3x + 5$, *then* $\frac{dy}{dx} = f'(x) = 18x^2 + 4x + 3$.

5.4.1 Economic Applications

In general, if our original function represents a total function (e.g. total cost, total revenue, etc.), then its derivative is its marginal function (e.g. marginal cost, marginal revenue, etc.).

Example 11 Marginal cost

Suppose a firm faces the following total cost function

$$C(Q) = Q^3 + 4Q^2 + 10Q + 75$$

Then marginal cost is given by

$$MC = \frac{dC}{dQ} = 3Q^2 + 8Q + 10$$

In general form,

$$\begin{aligned} \textit{If } C &= C(Q) \\ \textit{then } MC &= \frac{dC}{dQ} = C'(Q) \end{aligned}$$

Example 12 Marginal revenue

In general marginal revenue is the derivative of the total revenue function

$$\begin{aligned} \textit{If } R &= R(Q) \\ \textit{then } MR &= \frac{dR}{dQ} = R'(Q) \end{aligned}$$

Suppose a monopolist faces the demand function

$$Q = 27 - 3P$$

Then total revenue, expressed as a function of Q is

$$\begin{aligned} R(Q) = PQ &= \left(\frac{1}{3}(27 - Q)\right)(Q) \\ &= 9Q - \frac{1}{3}Q^2 \end{aligned}$$

Marginal revenue is then

$$MR = \frac{dR}{dQ} = R'(Q) = 9 - \frac{2}{3}Q$$

Example 13 Marginal propensity to consume

Suppose we have the consumption function

$$C = 10 + 0.7Y - 0.002Y^2$$

The marginal propensity to consume is

$$\frac{dC}{dY} = 0.7 - 0.004Y$$

5.5 Product Rule

$$\begin{aligned} \text{If } y &= f(x)\,g(x) \\ \text{then } \frac{dy}{dx} &= \frac{d}{dx}[f(x)\,g(x)] \\ &= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \\ &= f(x)\,g'(x) + g(x)\,f'(x) \end{aligned}$$

Example 14

$$
\begin{aligned}
& \frac{d}{dx}\left[\left(4x^3+5\right)\left(3x^2-8\right)\right] \\
= & \left(4x^3+5\right)\frac{d}{dx}\left[\left(3x^2-8\right)\right]+\left(3x^2-8\right)\frac{d}{dx}\left[\left(4x^3+5\right)\right] \\
= & \left(4x^3+5\right)(6x)+\left(3x^2-8\right)\left(12x^2\right) \\
= & 24x^4+30x+36x^4-96x^2 \\
= & 60x^4-96x^2+30x
\end{aligned}
$$

Example 15

$$
\begin{aligned}
& \frac{d}{dx}\left[(5x+2)\left(2x^2\right)\right] \\
= & (5x+2)(4x)+\left(2x^2\right)(5) \\
= & 30x^2+8x
\end{aligned}
$$

5.5.1 Economic Application

Example 16 Finding the marginal revenue function from the average revenue function

Suppose a firm faces an average revenue function $AR = 20 - Q$.

(We know from economic theory, that the average revenue function is a function of output ($AR = f(Q)$). Recall that $AR = P$ ($AR \equiv \frac{TR}{Q} \equiv \frac{PQ}{Q} \equiv P$). So $P = f(Q)$, i.e. the average revenue curve is the inverse *of the demand curve.)*

So, to find the marginal revenue function, first calculate total revenue:

$$
\begin{aligned}
TR = P \times Q &= AR \times Q \\
&= (20-Q)\,Q \\
&= 20Q - Q^2
\end{aligned}
$$

Marginal revenue is given by the slope of the total revenue curve, so we simply find the derivative of the total revenue function:

$$
MR = \frac{dTR}{dQ} = 20 - 2Q
$$

Example 17 *In general form*

$$
\begin{aligned}
AR &= f(Q) \\
TR &= AR \times Q = f(Q) \times Q \\
MR &= \frac{dTR}{dQ} = f'(Q)\,Q + f(Q)
\end{aligned}
$$

Recall that $f(Q) = AR$*, so*

$$\begin{aligned} MR &= f'(Q)\,Q + AR \\ &\Rightarrow MR - AR = Qf'(Q) \end{aligned}$$

Thus, MR *and* AR *will always differ by* $Qf'(Q)$*.*
Let us evaluate this result:
Q *is quantity and so is always positive.*
$f'(Q)$ *is the slope of the* AR *curve.*

- *Under perfect competition, the* AR *curve is a horizontal straight line (because all firms are price takers). Therefore* $f'(Q) = 0$*, and*

$$\begin{aligned} MR - AR &= 0 \\ \Rightarrow MR &= AR \end{aligned}$$

  Thus, under perfect competition the MR *curve and* AR *curve coincide.*

- *Under imperfect competition, however, the* AR *curve is downward sloping. Therefore,* $f'(Q) < 0$*, and*

$$\begin{aligned} MR - AR &< 0 \\ \Rightarrow MR &< AR \end{aligned}$$

  Thus, under imperfect competition the MR *curve lies below the* AR *curve.*

5.6 Quotient Rule

$$\begin{aligned} \text{If } y &= \frac{f(x)}{g(x)} \\ \text{then } \frac{dy}{dx} &= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2} \end{aligned}$$

Example 18

$$\begin{aligned} \frac{d}{dx}\left[\frac{2x-3}{x+1}\right] &= \frac{(2)(x+1) - (2x-3)(1)}{(x+1)^2} \\ &= \frac{5}{(x+1)^2} \end{aligned}$$

Example 19

$$\begin{aligned}\frac{d}{dx}\left[\frac{5x+2}{x^2-2x+1}\right] &= \frac{(5)\left(x^2-2x+1\right)-(5x+2)(2x-2)}{\left(x^2-2x+1\right)^2} \\ &= \frac{-5x^2-4x+9}{\left(x^2-2x+1\right)^2} \\ &= \frac{-(5x+9)(x-1)}{(x-1)^4} \\ &= \frac{-(5x+9)}{(x-1)^3}\end{aligned}$$

Example 20

$$\begin{aligned}\frac{d}{dx}\left[\frac{ax^2+b}{cx}\right] &= \frac{(2ax)(cx)-\left(ax^2+b\right)(c)}{(cx)^2} \\ &= \frac{2acx^2-acx^2-bc}{c^2x^2} \\ &= \frac{acx^2-bc}{c^2x^2} \\ &= \frac{c\left(ax^2-b\right)}{c^2x^2} \\ &= \frac{ax^2-b}{cx^2}\end{aligned}$$

5.6.1 Economic Application

Example 21 The relationship between the marginal cost and average cost

Recall

$$\begin{aligned}\textit{Total cost:} \quad C &= C(Q) \\ \textit{Average cost:} \quad AC &= \frac{C(Q)}{Q}\end{aligned}$$

The slope of the AC curve can be found by finding its derivative

$$\begin{aligned}\frac{dAC}{dQ} &= \frac{C'(Q)\,Q-C(Q)(1)}{Q^2} \\ &= \frac{C'(Q)}{Q}-\frac{C(Q)}{Q^2} \\ &= \frac{1}{Q}\left[C'(Q)-\frac{C(Q)}{Q}\right] \\ &= \frac{1}{Q}\left[MC-AC\right]\end{aligned}$$

We can use this result to tell us where the MC curve intersects the AC curve. (Assume that $Q > 0$)

$$\begin{aligned}
\frac{dAC}{dQ} &> 0 \text{ if } C'(Q) > \frac{C(Q)}{Q} \\
\frac{dAC}{dQ} &= 0 \text{ if } C'(Q) = \frac{C(Q)}{Q} \\
\frac{dAC}{dQ} &< 0 \text{ if } C'(Q) < \frac{C(Q)}{Q}
\end{aligned}$$

In words, the slope of the AC curve is

- *positive if MC lies above AC*
- *zero if MC intersects AC*
- *negative if MC lies below AC*

This establishes the familiar result that the MC curve intersects the AC curve at its minimum point.

5.7 Chain Rule

This is useful when we have a composite function.

$$\begin{aligned}
\text{If } f(x) &= p(q(x)) \\
\text{then } f'(x) &= p'(q(x))\, q'(x)
\end{aligned}$$

A simpler way of writing this is as follows. Let $u = q(x)$ and $y = p(u)$, so that $y = p(q(x)) = f(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example 22

$$f(x) = \left(5x^2 - 1\right)^9$$

Let $u = 5x^2 - 1$, and $y = u^9$. Then

$$\begin{aligned}
f'(x) = \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
&= \left(9u^8\right) \times (10x) \\
&= (10x)\left(9\left(5x^2 - 1\right)^8\right) \\
&= 90x\left(5x^2 - 1\right)^8
\end{aligned}$$

Example 23

$$f(x) = 3(2x+5)^2$$

Let $u = 2x+5$, and $y = 3u^2$. Then

$$\begin{aligned} f'(x) = \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= (6u) \times (2) \\ &= 12(2x+5) \\ &= 24x + 60 \end{aligned}$$

Example 24

$$f(x) = \left(x^2 + 3x - 2\right)^{17}$$

Let $u = x^2 + 3x - 2$, and $y = u^{17}$. Then

$$\begin{aligned} f'(x) = \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \left(17u^{16}\right) \times (2x+3) \\ &= 17\left(x^2 + 3x - 2\right)^{16}(2x+3) \end{aligned}$$

5.7.1 Economic Application

Example 25 *Given the total revenue function of a firm $R = f(Q)$, where output Q is a function of labour input L $(Q = g(L))$, find $\frac{dR}{dL}$.*

$$\begin{aligned} R &= f(Q) = f(g(L)) \\ \frac{dR}{dL} &= \frac{dR}{dQ}.\frac{dQ}{dL} = f'(Q)\,g'(L) \end{aligned}$$

In economic terms $\frac{dR}{dL}$ is the marginal revenue product of labour (MRP_L), $f'(Q)$ is the marginal revenue function (MR), and $g'(L)$ is the marginal physical product of labour (MPP_L). Thus our result gives us the well-known economic relationship

$$MRP_L = MR \times MPP_L.$$

5.8 Inverse Function Rule

Definition 5 *If the function $y = f(x)$ represents a one-to-one mapping, i.e. if the function is such that each value of y is associated with a unique value of x, the function f will have an* ***inverse function*** *$x = f^{-1}(y)$. Note this is not the reciprocal of $f(x)$ (i.e. $f^{-1}(y) \neq \frac{1}{f(x)}$).*

When an inverse function exists, this means that every x value will yield a unique y value, and every y value will yield a unique x value, i.e. there is a *one-to-one mapping.*

Example 26 *The mapping from the set of all husbands to the set of all wives is one-to-one: each husband has a unique wife and each wife has a unique husband (in a monogamous society).*

The mapping from the set of all fathers to the set of all sons is not one-to-one: each father may have more than one son, although each son has a unique father.

When x and y refer specifically to numbers, the property of one-to-one mapping is unique to strictly monotonic functions.

Definition 6 *Given a function $f(x)$, if*

$$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$$

then f is said to be a ***strictly increasing*** *function. If*

$$x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$$

then f is said to be a ***strictly decreasing*** *function.*

In either of these cases, f is said to be a ***strictly monotonic*** *function, and an inverse function f^{-1} exists.*

A practical way of determining whether a function $f(x)$ is strictly monotonic is to check whether the derivative $f'(x)$ is either always positive or always negative (not zero) for all values of x. So

$$\begin{aligned} \text{If } f'(x) &> 0\ \forall x, \text{ then } f(x) \text{ is strictly increasing (upward sloping).} \\ \text{If } f'(x) &< 0\ \forall x, \text{ then } f(x) \text{ is strictly decreasing (downward sloping).} \end{aligned}$$

Example 27

$$\begin{aligned} y &= 5x + 25 \\ \frac{dy}{dx} &= 5 \end{aligned}$$

Since $\frac{dy}{dx}$ is positive regardless of the value of x, this function is strictly increasing. It follows that it is monotonic and an inverse function exists. In this case, we can easily find the inverse function by solving the equation $y = 5x + 25$ for x

$$x = \frac{1}{5}y - 5$$

Example 28 *Show that the total cost function* $TC = Q^3 - 53Q^2 + 940Q + 1500$ *is strictly monotonic.*

Find the first derivative:

$$TC'(Q) = \frac{dTC}{dQ} = 3Q^2 - 106Q + 940$$

Show that this is strictly positive:

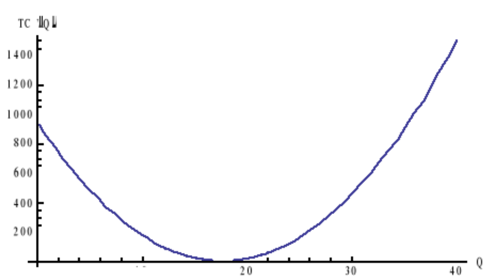
1) The coefficient on Q^2 *is positive, so the parabola will be convex. Thus, if the parabola has no x-intercepts we know it will lie strictly above the x-axis, i.e.* $TC'(Q) > 0$, $\forall Q$.

$$\begin{aligned} b^2 - 4ac &= (-106)^2 - 4(3)(940) \\ &= -44 \\ &< 0 \Rightarrow \text{ no } x\text{-intercepts.} \end{aligned}$$

2) OR, by completing the square (see Section 0, p11-12):

$$\begin{aligned} TC'(Q) &= \frac{dTC}{dQ} = 3Q^2 - 106Q + 940 \\ &= 3\left(Q^2 - \frac{106}{3}Q + \frac{940}{3}\right) \\ &= 3\left(Q^2 - \frac{106}{3}Q + \frac{2809}{9} - \frac{2809}{9} + \frac{940}{3}\right) \\ &= 3\left[\left(Q - \frac{53}{3}\right)^2 + \frac{11}{9}\right] \\ &> 0 \text{ for all } Q \end{aligned}$$

3) OR, by graphing it:



For inverse functions, the rule of differentiation is

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

Example 29 *Given* $y = x^5 + x$, *find* $\frac{dx}{dy}$.

First, we need to determine whether an inverse function exists.

$$\frac{dy}{dx} = 5x^4 + 1$$

Since $\frac{dy}{dx}$ *is positive regardless of the value of* x, *this function is strictly increasing. It follows that it is monotonic and an inverse function exists. In this case, it is not so easy to solve the given equation for* x, *but we can easily find the derivative of the inverse function using the inverse function rule:*

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{5x^4 + 1}$$

5.9 Log Function Rule

$$\begin{aligned} \text{If } y &= f(x) = \log_b x \\ \text{then } \frac{dy}{dx} &= f'(x) = \frac{1}{x \ln b} \end{aligned}$$

For natural logarithms (base e), the rule becomes

$$\begin{aligned} \text{If } y &= f(x) = \ln x \\ \text{then } \frac{dy}{dx} &= f'(x) = \frac{1}{x \ln e} = \frac{1}{x} \end{aligned}$$

Example 30 *If* $y = f(x) = 5 \ln x$, *then* $\frac{dy}{dx} = f'(x) = \frac{5}{x}$.

Example 31 *If* $y = f(x) = \frac{\ln x}{x^2}$, *then*

$$\begin{aligned} \frac{dy}{dx} = f'(x) &= \frac{(1/x)\left(x^2\right) - (\ln x)(2x)}{\left(x^2\right)^2} \\ &= \frac{1 - 2\ln x}{x^3} \end{aligned}$$

In some cases, our logarithmic function may be a little more complex. For example, the x in our function $\ln x$, may be replaced by some function of x, $g(x)$. In this case, we use the chain rule (let $u = g(x)$, then $y = \ln u$)

$$\begin{aligned} \text{If } y &= \ln g(x) \\ \text{then } \frac{dy}{dx} &= \frac{g'(x)}{g(x)} \end{aligned}$$

Example 32 *If* $y = \ln\left(x^2+1\right)$, *then*

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{2x}{x^2+1}$$

Example 33 *If* $y = \ln\left(x^5+2\right)$, *then*

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{5x^4}{x^5+2}$$

Example 34 *If* $y = \ln\left(2x^2+3x\right)$, *then*

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{4x+3}{2x^2+3x}$$

Example 35 *If* $y = x^2 \ln(4x+2)$, *then*

$$\begin{aligned} \frac{dy}{dx} &= \left(x^2\right)\left(\frac{4}{4x+2}\right) + (2x)(\ln(4x+2)) \\ &= \frac{2x^2}{2x+1} + 2x\ln(4x+2) \end{aligned}$$

5.10 Exponent Function Rule

$$\begin{aligned} \text{If } y &= f(x) = e^x \\ \text{then } \frac{dy}{dx} &= f'(x) = e^x \end{aligned}$$

In those cases where the x is replaced by some function of x, $g(x)$, we use the chain rule (let $u = g(x)$, then $y = e^u$):

$$\begin{aligned} \text{If } y &= f(x) = e^{g(x)} \\ \text{then } \frac{dy}{dx} &= f'(x) = g'(x)\, e^{g(x)} \end{aligned}$$

Example 36 *If* $y = e^{-1/2x^2}$, *then* $\frac{dy}{dx} = -xe^{-1/2x^2}$.

Example 37 *If* $y = e^{rt}$, *then* $\frac{dy}{dt} = re^{rt}$.

Example 38 *If* $y = e^{-t}$, *then* $\frac{dy}{dt} = -e^{-t}$.

6 Logarithmic Differentiation

Often, the functional forms one is presented with are incredibly complex. One way of simplifying the task of differentiation is to re-write the function in natural logarithms before finding the derivative. In general, to differentiate any function $y = f(x)$ (but especially complicated ones that entail using the product, quotient and power rules all at once), the following method might be useful:

1. Take the natural log of both sides, thus obtaining $\ln y = \ln[f(x)]$
2. Simplify $\ln[f(x)]$ by using properties of logs.
3. Differentiate both sides with respect to x.
4. Solve for $\frac{dy}{dx}$.
5. Re-write in terms of x only.

Example 39 *If* $y = \frac{(2x-5)^2}{x^2\sqrt[4]{x^2+1}}$, *then to find* $\frac{dy}{dx}$:

Step 1*: Take the natural log of both sides*

$$\ln y = \ln\left[\frac{(2x-5)^2}{x^2\sqrt[4]{x^2+1}}\right]$$

Step 2*: Simplify the RHS by using properties of logs*

$$\begin{aligned}
\ln y &= \ln(2x-5)^2 - \ln\left(x^2\sqrt[4]{x^2+1}\right) \\
\ln y &= 2\ln(2x-5) - \left(\ln x^2 + \ln\left(x^2+1\right)^{1/4}\right) \\
\ln y &= 2\ln(2x-5) - 2\ln x - \frac{1}{4}\ln\left(x^2+1\right)
\end{aligned}$$

Step 3*: Differentiate BOTH sides with respect to x.*
On the LHS, we use the chain rule. Let $z = \ln y$, where $y = f(x)$. Then

$$\frac{dz}{dx} = \frac{dz}{dy}.\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

On the RHS

$$\begin{aligned} & \frac{d}{dx}\left[2\ln(2x-5) - 2\ln x - \frac{1}{4}\ln\left(x^2+1\right)\right] \\ = \quad & \frac{4}{2x-5} - \frac{2}{x} - \frac{x}{2\left(x^2+1\right)} \end{aligned}$$

Now LHS=RHS

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{2x-5} - \frac{2}{x} - \frac{x}{2\left(x^2+1\right)}$$

Step 4*: Solve for $\frac{dy}{dx}$.*

$$\frac{dy}{dx} = y\left[\frac{4}{2x-5} - \frac{2}{x} - \frac{x}{2\left(x^2+1\right)}\right]$$

Step 5*: Re-write in terms of x only*

$$\frac{dy}{dx} = \frac{(2x-5)^2}{x^2\sqrt[4]{x^2+1}}\left[\frac{4}{2x-5} - \frac{2}{x} - \frac{x}{2\left(x^2+1\right)}\right]$$

7 Partial Differentiation

So far we have only looked at situations in which there was only one independent variable (we call such functions "bivariate" functions). However, most interesting applications in science (be it in the social, physical, behavioural or biological sciences) require an analysis of how one variable changes with infinitesimal changes in another, when there is more than one independent variable. Such functions are called *multivariate* functions. A simple example is $f(x, y) = x^2 + y^2$. Another would be $Q = Q(L, K)$. The first is a numerical function as it specifies an analytical expression in the two variables, whereas the second is a general function - it does not tell you the explicit functional relation of L and K with respect to Q, just that these are the two variables that explain the dependant variable Q.

So if $y = f(x_1, x_2, \ldots, x_n)$, when we partially differentiate y with respect to x_i we allow x_i to vary and hold the other independent variables constant. We denote the **partial derivative** of y with respect to x_i by

$$\frac{\partial y}{\partial x_i} = f_{x_i}$$

This is the ceteris paribus assumption you have encountered before: what is the effect of x_i on y, holding other things constant?

Since we already know how to handle constants when dealing with just one independent variable, you in effect already know how to partially differentiate a function.

Example 40 *If* $y = f(x_1, x_2) = 3x_1^2 + x_1 x_2 + 4x_2^2$, *then*

$$\begin{aligned} \frac{\partial y}{\partial x_1} &= f_{x_1} = 6x_1 + x_2 && \text{We treat } x_2 \text{ as constant, and allow } x_1 \text{ to vary.} \\ \frac{\partial y}{\partial x_2} &= f_{x_2} = x_1 + 8x_2 && \text{We treat } x_1 \text{ as constant, and allow } x_2 \text{ to vary.} \end{aligned}$$

Example 41 *If* $y = f(u, v) = (u + 4)(3u + 2v)$, *then*

$$\begin{aligned} \frac{\partial f}{\partial u} &= f_u = (u+4)(3) + (1)(3u+2v) = 6u + 2v + 12 = 2(3u + v + 6) \\ \frac{\partial f}{\partial v} &= f_v = (u+4)(2) + (0)(3u+2v) = 2(u+4) \end{aligned}$$

Example 42 *If* $y = f(u, v) = \dfrac{3u - 2v}{u^2 + 3v}$, *then*

$$\begin{aligned} \frac{\partial f}{\partial u} &= f_u = \frac{(3)\left(u^2+3v\right) - (3u-2v)(2u)}{\left(u^2+3v\right)^2} = \frac{-3u^2 + 4uv + 9v}{\left(u^2+3v\right)^2} \\ \frac{\partial f}{\partial v} &= f_v = \frac{(-2)\left(u^2+3v\right) - (3u-2v)(3)}{\left(u^2+3v\right)^2} = \frac{-u(2u+9)}{\left(u^2+3v\right)^2} \end{aligned}$$

Example 43 *Suppose we have a production function given by* $Q = f(K, L)$. *Then we can find the partial derivatives with respect to* K *and* L, *which have particular meanings:*

$$\frac{\partial Q}{\partial K} = Q_K = f_K = \text{ marginal product of capital}$$

This tells us how output will vary in response to a one-unit change in capital input, holding the labour input constant. Similarly,

$$\frac{\partial Q}{\partial L} = Q_L = f_L = \text{ marginal product of labour}$$

This tells us how output will vary in response to a one-unit change in labour input, holding the capital input constant.

7.1 Applications to Comparative Static Analysis

We analyse the comparative statics of the equilibrium of economic models that we've already solved.

Example 44 ***Market Model***

This model is the familiar supply and demand framework in a market producing 1 good. We will now use calculus to look at how changes in the intercept and slopes of demand and supply functions affect equilibrium *price and quantity The emphasis on the word* equilibrium *is important here as it indicates that what we interested in is how the solution to the demand and supply functions changes when you change one of the parameters of the model. (i.e. comparative static analysis).*

$$\begin{aligned} Q_d &= Q_s \\ Q_d &= a - bP \qquad (a, b > 0) \\ Q_s &= -c + dP \qquad (c, d > 0) \end{aligned}$$

We've solved for the equilibrium price and quantity in Section 1:

$$\begin{aligned} P^* &= \frac{a+c}{b+d} \\ Q^* &= \frac{ad-bc}{b+d} \end{aligned}$$

We are now interested in analysing the comparative statics of the model.

$$\frac{\partial P^*}{\partial a} = \frac{1}{b+d} > 0$$

This tells us P^* will increase (decrease) if a increases (decreases).

$$\frac{\partial P^*}{\partial b} = \frac{0\,(b+d) - 1\,(a+c)}{(b+d)^2} = \frac{-\,(a+c)}{(b+d)^2} < 0$$

This tells us P^* will decrease (increase) if b increases (decreases).

$$\frac{\partial P^*}{\partial c} = \frac{1}{b+d} > 0$$

This tells us P^* will increase (decrease) if c increases (decreases).

$$\frac{\partial P^*}{\partial d} = \frac{0\,(b+d) - 1\,(a+c)}{(b+d)^2} = \frac{-\,(a+c)}{(b+d)^2} < 0$$

This tells us P^* will decrease (increase) if d increases (decreases).
Find the partial derivatives of Q^* and check your results using graphs.

Example 45 ***National Income Model***

Let the national income model be:

$$\begin{aligned} Y &= C + I + G \\ C &= a + b(Y - T) & a > 0, 0 < b < 1 \\ T &= d + tY & d > 0, 0 < t < 1 \end{aligned}$$

where Y is national income, C is (planned) consumption expenditure, I is investment expenditure, G is government expenditure and T is taxes.

We've solved for the equilibrium in Section 2:

$$\begin{aligned} Y^* &= \frac{a - bd + I + G}{1 - b(1-t)} \\ C^* &= \frac{a - bd + b(1-t)(I+G)}{1 - b(1-t)} \\ T^* &= \frac{d(1-b) + t(a + I + G)}{1 - b(1-t)} \end{aligned}$$

Government multiplier:

$$\frac{\partial Y^*}{\partial G} = \frac{1}{1 - b(1-t)} > 0$$

An increase (decrease) in government expenditure will increase (decrease) equilibrium national income.

The effect of a change in non-income tax on equilibrium national income:

$$\frac{\partial Y^*}{\partial d} = \frac{-b}{1 - b(1-t)} < 0$$

An increase (decrease) in non-income tax will decrease (increase) equilibrium national income.

The effect of a change in income tax on equilibrium national income:

$$\frac{\partial Y^*}{\partial t} = \frac{0(1 - b(1-t)) - (a - bd + I + G)(b)}{(1 - b(1-t))^2} = \frac{-bY^*}{1 - b(1-t)} < 0$$

An increase (decrease) in income tax will decrease (increase) equilibrium national income.

You should always check that your results coincide with your economic intuition.

Example 46 ***Market Model***

The system of equations below describes the market for widgets:

$$\begin{aligned} Q_d &= \alpha - \beta P + \gamma G \\ Q_s &= -\delta + \theta P - \lambda N \\ Q_d &= Q_s \qquad\qquad \alpha, \beta, \gamma, \delta, \theta, \lambda > 0 \end{aligned}$$

where G is the price of substitutes for widgets and N is the price of inputs used in producing widgets.

We've solved for the equilibrium in Section 2:

$$\begin{aligned} Q^* &= \frac{\theta(\alpha + \gamma G) - \beta(\delta + \lambda N)}{(\beta + \theta)} \\ P^* &= \frac{\delta + \lambda N + \alpha + \gamma G}{(\beta + \theta)} \end{aligned}$$

Show how an increase in the price of substitute goods, G, affects equilibrium quantity and price.

$$\frac{\partial Q^*}{\partial G} = \frac{\theta\gamma}{(\beta + \theta)} > 0$$

An increase in the price of substitutes will increase equilibrium quantity.

$$\frac{\partial P^*}{\partial G} = \frac{\gamma}{(\beta + \theta)} > 0$$

An increase in the price of substitutes will increase equilibrium price.

Show how an increase in the price of inputs, N, affects equilibrium quantity and price.

$$\frac{\partial Q^*}{\partial N} = \frac{-\beta\lambda}{(\beta + \theta)} < 0$$

An increase in the price of inputs will reduce equilibrium quantity.

$$\frac{\partial P^*}{\partial N} = \frac{\lambda}{(\beta + \theta)} > 0$$

An increase in the price of inputs will increase equilibrium price.

7.2 Jacobian Determinants

Partial derivatives give us a way to test whether **functional dependence** (linear or non-linear) exists among a set of n functions in n variables.

Consider the case of two functions in two variables:

$$\begin{aligned} y_1 &= f(x_1, x_2) \\ y_2 &= g(x_1, x_2) \end{aligned}$$

We can take all four partial derivatives for these functions, and arrange them in a square matrix in a specific order. This matrix, denoted by a J, is called a *Jacobian matrix.*

$$J = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 \end{bmatrix}$$

If the determinant of this Jacobian matrix is zero, this indicates that there is functional dependence among our functions.

$$\begin{aligned} |J| &= \left(\frac{\partial y_1}{\partial x_1}\right)\left(\frac{\partial y_2}{\partial x_2}\right) - \left(\frac{\partial y_1}{\partial x_2}\right)\left(\frac{\partial y_2}{\partial x_1}\right) \\ &= 0 \Rightarrow \text{functional dependence} \\ &\neq 0 \Rightarrow \text{functional independence} \end{aligned}$$

Why use a Jacobian? The advantage of using a Jacobian determinant is that it allows us to detect linear and non-linear dependence in functions. Up until now, in our matrix algebra, we have only been able to detect linear dependence in a system of linear functions.

Example 47

$$\begin{aligned} y_1 &= 2x_1 + 3x_2 \\ y_2 &= 4x_1^2 + 12x_1x_2 + 9x_2^2 \end{aligned}$$

The Jacobian matrix is

$$\begin{aligned} J &= \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \\ 8x_1 + 12x_2 & 12x_1 + 18x_2 \end{bmatrix} \end{aligned}$$

and the Jacobian determinant is

$$\begin{aligned} |J| &= 2(12x_1 + 18x_2) - 3(8x_1 + 12x_2) \\ &= 0 \end{aligned}$$

This means that there is functional dependence among our functions (in particular $y_2 = (y_1)^2$*).*

8 General-Function Models

In all the models we've dealt with so far, we've been able to solve for the reduced form equations. This has allowed us to use partial differentiation to figure out the comparative statics of our model. But to be able to use partial differentiation in the first place, a *requirement* is that there must be *functional independence among independent variables.* For example, if we have:

$$y = f(x_1, x_2) = 3x_1 + 4x_2$$

then, to use partial differentiation, a key requirement is that x_1 when changes, x_2 remains constant. In other words, there is no functional relationship between x_1 and x_2 that would cause x_2 to change when x_1 changes. This means that the parameters or exogenous variables that appear in the reduced form equation are mutually independent.

Suppose instead that we have a function $y = f(x_1, x_2)$ but x_2 changes when x_1 changes and vice versa (i.e. there is interdependence). In this case we can no longer use partial differentiation.

Often we have models that contain general functions, which mean that we cannot actually explicitly solve for the reduced form. For example, consider the simple national income model

$$\begin{aligned} Y &= C + I_0 + G_0 \\ C &= C(Y, T_0) \end{aligned}$$

which can be written as a single equation (an equilibrium condition)

$$Y = C(Y, T_0) + I_0 + G_0$$

to be solved for Y^*. Because the C function is given in general form, we cannot find an explicit solution for Y^*. So, we will have to find the comparative static derivatives directly from this function.

Let us suppose that Y^* exists. Then the following identity will hold:

$$Y^* \equiv C(Y^*, T_0) + I_0 + G_0$$

It may seem that simple partial differentiation of this identity will give us any desired comparative static derivative, say $\dfrac{\partial Y^*}{\partial T_0}$. Unfortunately this is not the case. Since Y^* is a function of T_0, the two arguments of the consumption function are no longer independent. T_0 can affect consumption *directly*, but also *indirectly* through its effect on Y^*.

The minute this type of interdependence arises, we can no longer use partial differentiation. Instead, we use **total differentiation**.

9 Differentials and Derivatives

So far, we know that a derivative for a function $y = f(x)$ can be represented as $\frac{dy}{dx}$. We now re-interpret this as a ratio of two quantities dy and dx. Think of dy as the infinitesimal change in y, and dx as the infinitesimal change in x.

$$\begin{aligned} \frac{dy}{dx} &\equiv \frac{dy}{dx} \\ \therefore dy &= \left(\frac{dy}{dx}\right) dx = f'(x)\, dx \end{aligned}$$

The derivative $f'(x)$ can then be reinterpreted as the factor of proportionality between the two finite changes dy and dx. Accordingly given a specific value of dx, we can multiply it by $f'(x)$ to get dy. The quantities dy and dx are called the **differentials** of x and y, respectively.

Example 48

$$\begin{aligned} \text{If } y &= 3x^2 + 7x - 5 \\ \text{then } dy &= f'(x)\, dx \\ &= (6x + 7)\, dx \end{aligned}$$

Example 49

$$\begin{aligned} \text{If } y &= 10x^3 + 2x^2 - 5x + 1 \\ \text{then } dy &= f'(x)\, dx \\ &= \left(30x^2 + 4x - 5\right) dx \end{aligned}$$

9.1 Differentials and Point Elasticity

Given a demand function $Q = f(P)$, its elasticity is defined as

$$\frac{\Delta Q/Q}{\Delta P/P} = \frac{\Delta Q/\Delta P}{Q/P}$$

If the change in P is infinitesimal, then the expressions ΔQ and ΔP reduce to the differentials dP and dQ. We can re-write our expression as the *point elasticity* of demand:

$$\varepsilon_d \equiv \frac{dQ/dP}{Q/P}$$

Now look at the numerator of this expression: dQ/dP is the derivative, or the *marginal* function (slope), of the demand function $Q = f(P)$.

Now look at the denominator: Q/P which is the *average* function of the demand function. In other words, the point elasticity of demand ε_d is the ratio of the marginal function to the average function of the demand function.

This is valid for any other function too. For any given *total* function $y = f(x)$ we can write the point elasticity of y with respect to x as

$$\varepsilon_{yx} = \frac{dy/dx}{y/x} = \frac{\text{marginal function}}{\text{average function}}$$

By convention, the absolute value of the elasticity measure is used in deciding whether the function is elastic at a particular point. For instance, for demand functions we say

$$\text{Demand is } \left\{ \begin{array}{c} \text{elastic} \\ \text{of unit elasticity} \\ \text{inelastic} \end{array} \right\} \text{ at a point when } |\varepsilon_d| \gtreqless 1.$$

Example 50 *Find ε_d if the demand function is $Q = 200 - 4P$.*

$$\frac{dQ}{dP} = -4 \text{ and } \frac{Q}{P} = \frac{200 - 4P}{P}$$

$$\begin{aligned} \therefore \varepsilon_d = \frac{dQ/dP}{Q/P} &= \frac{-4}{(200 - 4P)/P} \\ &= -4 \times \frac{P}{200 - 4P} \\ &= \frac{-4P}{200 - 4P} \\ &= \frac{-P}{50 - P} \end{aligned}$$

This solution as it stands is written as a function of P. However, should you be given a value of P (e.g. $P = 25$), you could then explicitly solve for the elasticity at that price. In this case, if $P = 25$ the elasticity of demand would be -1, in other words, demand elasticity is unitary at that point ($|\varepsilon_d| = 1$).

10 Total Differentials

We now extend the idea of a differential to a function that has more than one independent variable. For example, if

$$y = f(x_1, x_2)$$

then the easiest way to proceed might be to find the two separate partial derivatives f_{x_1} and f_{x_2}, and then substitute these into the equation:

$$\begin{aligned} dy &= \frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2 \\ &= \underbrace{f_{x_1}dx_1}_{\substack{\text{change in } y \\ \text{due to change in } x_1}} + \underbrace{f_{x_2}dx_2}_{\substack{\text{change in } y \\ \text{due to change in } x_2}} \end{aligned}$$

dy is called the **total differential** of the y function. It is the sum of the change that occurs from a change in x_1 and x_2. The process of finding total differentials is called *total differentiation.*

Example 51

$$\begin{aligned} \textit{If } z &= 3x^2 + xy - 2y^3 \\ \textit{then } dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= (6x + y)\,dx + \left(x - 6y^2\right)dy \end{aligned}$$

Example 52

$$\begin{aligned} \textit{If } U &= 2x_1 + 9x_1x_2 + x_2^2 \\ \textit{then } dU &= \frac{\partial U}{\partial x_1}dx_1 + \frac{\partial U}{\partial x_2}dx_2 \\ &= (2 + 9x_2)\,dx_1 + (9x_1 + 2x_2)\,dx_2 \end{aligned}$$

Example 53

$$\begin{aligned} \text{If } z &= \frac{x}{x+y} \\ \text{then } dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= \left(\frac{1(x+y) - x(1)}{(x+y)^2}\right)dx + \left(\frac{0(x+y) - x(1)}{(x+y)^2}\right)dy \\ &= \left(\frac{y}{(x+y)^2}\right)dx + \left(\frac{-x}{(x+y)^2}\right)dy \\ &= \left(\frac{y}{(x+y)^2}\right)dx - \left(\frac{x}{(x+y)^2}\right)dy \end{aligned}$$

Example 54

$$\begin{aligned} \text{If } y &= \frac{2xz}{x+z} \\ \text{then } dy &= \frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial z}dz \\ &= \left(\frac{2z(x+z) - 2xz(1)}{(x+z)^2}\right)dx + \left(\frac{2x(x+z) - 2xz(1)}{(x+z)^2}\right)dz \\ &= \left(\frac{2z^2}{(x+z)^2}\right)dx + \left(\frac{2x^2}{(x+z)^2}\right)dz \end{aligned}$$

10.1 Economic Applications

Example 55 *Consider a saving function*

$$S = S(Y, i)$$

where S is savings, Y is national income and i is the interest rate.

The total change in S is given by the differential

$$\begin{aligned} dS &= \frac{\partial S}{\partial Y}dY + \frac{\partial S}{\partial i}di \\ &= S_Y dY + S_i di \end{aligned}$$

The first term $S_Y dY$ gives the change in S resulting from the change in Y, and the second term $S_i di$ gives the change in S resulting from a change in i.

We can also find the elasticity of savings with respect to Y and i:

$$\begin{aligned} \varepsilon_{SY} &= \frac{\partial S/\partial Y}{S/Y} = S_Y \frac{Y}{S} \\ \varepsilon_{Si} &= \frac{\partial S/\partial i}{S/i} = S_i \frac{i}{S} \end{aligned}$$

Example 56 *Consider the utility function*

$$U = U(x_1, x_2)$$

The total differential of this function is

$$\begin{aligned} dU &= \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 \\ &= U_{x_1} dx_1 + U_{x_2} dx_2 \end{aligned}$$

Economically, the term $U_{x_1} dx_1$ *means the marginal utility of* x_1 (U_{x_1}) *times the change in the quantity of* x_1 *consumed* (dx_1). *Similarly for* $U_{x_2} dx_2$.

Again, we can find elasticity measures with respect to each argument in our function:

$$\begin{aligned} \varepsilon_{Ux_1} &= \frac{\partial U / \partial x_1}{U / x_1} = U_{x_1} \frac{x_1}{U} \\ \varepsilon_{Ux_2} &= \frac{\partial U / \partial x_2}{U / x_2} = U_{x_2} \frac{x_2}{U} \end{aligned}$$

Example 57 ***Using total differentials to find MRS***

Consider a utility function $\overline{U} = U(x_1, x_2)$. *We know that* $\overline{U}$ *will be constant along a given indifference curve.*

The total differential is given by

$$\begin{aligned} d\overline{U} &= \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 \\ &= U_{x_1} dx_1 + U_{x_2} dx_2 \end{aligned}$$

Because $\overline{U}$ *is constant,* $d\overline{U} = 0$. *Thus*

$$\begin{aligned} d\overline{U} = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 &= 0 \\ \Rightarrow \frac{\partial U}{\partial x_2} dx_2 &= -\frac{\partial U}{\partial x_1} dx_1 \\ \frac{dx_2}{dx_1} &= -\frac{\partial U / \partial x_1}{\partial U / \partial x_2} = MRS_{x_1 x_2} \end{aligned}$$

11 Rules of Differentials

A straightforward way of finding the total differential dy, given

$$y = f(x_1, x_2)$$

is to find the two separate partial derivatives f_{x_1} and f_{x_2}, and then substitute these into the equation:

$$\begin{aligned} dy &= \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 \\ &= f_{x_1} dx_1 + f_{x_2} dx_2 \end{aligned}$$

But at other times, various rules of differentials may be useful. These rules are very similar to the rules of differentiation.

Let k be a constant and u and v be two functions of the variables x_1 and x_2. Then we have the following rules:

1. $dk = 0$ (cf. constant function rule)
2. $d(cu^n) = cnu^{n-1} du$ (cf. power function rule)
3. $d(u \pm v) = du \pm dv$ (cf. sum-difference rule)
4. $d(uv) = vdu + udv$ (cf. product rule)
5. $d\left(\frac{u}{v}\right) = \frac{1}{v^2}(vdu - udv)$ (cf. quotient rule)

Example 58 *Find the total differential of the function*

$$y = 5x_1^2 + 3x_2$$

We can use the straightforward method

$$\begin{aligned} dy &= \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 \\ &= 10x_1 dx_1 + 3dx_2 \end{aligned}$$

Or we can let $u = 5x_1^2$ and $v = 3x_2$ and use the rules

$$\begin{aligned} dy &= d\left(5x_1^2\right) + d(3x_2) && \textit{(rule 3)} \\ &= 10x_1 dx_1 + 3dx_2 && \textit{(rule 2)} \end{aligned}$$

Example 59 *Find the total differential of the function*

$$y = 3x_1^2 + x_1 x_2^2$$

We can use the straightforward method

$$\begin{aligned} dy &= \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 \\ &= \left(6x_1 + x_2^2\right) dx_1 + \left(2x_1 x_2\right) dx_2 \end{aligned}$$

Or we can let $u = 3x_1^2$ and $v = x_1 x_2^2$ and use the rules

$$\begin{aligned} dy &= d\left(3x_1^2\right) + d\left(x_1 x_2^2\right) && \text{(rule 3)} \\ &= 6x_1 dx_1 + x_2^2 dx_1 + x_1 d\left(x_2^2\right) && \text{(rules 2 and 4)} \\ &= \left(6x_1 + x_2^2\right) dx_1 + \left(2x_1 x_2\right) dx_2 && \text{(rule 2)} \end{aligned}$$

Example 60 *Find the total differential of the function*

$$y = \frac{x_1 + x_2}{2x_1^2}$$

We can use the straightforward method

$$\begin{aligned} dy &= \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 \\ &= \left(\frac{1\left(2x_1^2\right) - \left(x_1 + x_2\right)\left(4x_1\right)}{4x_1^4}\right) dx_1 + \left(\frac{1\left(2x_1^2\right) - \left(x_1 + x_2\right)(0)}{4x_1^4}\right) dx_2 \\ &= \left(\frac{-2x_1^2 - 4x_1 x_2}{4x_1^4}\right) dx_1 + \left(\frac{2x_1^2}{4x_1^4}\right) dx_2 \\ &= \left(\frac{-\left(x_1 + 2x_2\right)}{2x_1^3}\right) dx_1 + \left(\frac{1}{2x_1^2}\right) dx_2 \end{aligned}$$

Or we can let $u = x_1 + x_2$ and $v = 2x_1^2$ and use the rules

$$\begin{aligned} dy &= \frac{1}{\left(2x_1^2\right)^2} \left[2x_1^2 d\left(x_1 + x_2\right) - \left(x_1 + x_2\right) d\left(2x_1^2\right)\right] && \text{(rule 5)} \\ &= \frac{1}{4x_1^4} \left[2x_1^2 \left(dx_1 + dx_2\right) - \left(x_1 + x_2\right) 4x_1 dx_1\right] && \text{(rules 2 and 3)} \\ &= \frac{1}{4x_1^4} \left[-2x_1 \left(x_1 + 2x_2\right) dx_1 + 2x_1^2 dx_2\right] \\ &= \left(\frac{-\left(x_1 + 2x_2\right)}{2x_1^3}\right) dx_1 + \left(\frac{1}{2x_1^2}\right) dx_2 \end{aligned}$$

12 Total Derivatives

Now that we know how to find total differentials, we are closer to being able to figure out how to differentiate a function when the arguments of the function are not independent. Returning to our earlier example, we are a step closer to being able to find the derivative of the function $C(Y^*, T_0)$ with respect to T_0, when Y^* and T_0 are interrelated. To do this, we need to make use the **total derivative**. A total derivative does not require that Y^* remain constant as T_0 varies. In other words, a total derivative allows us to figure out the rate of change of a function written in general form, when the arguments in that function are related.

So how do we find the total derivative?

The total derivative is just a ratio of two differentials.

Step 1: Find the total differential

Step 2: Divide by the relevant differential

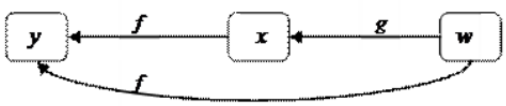
Suppose we have

$$y = f(x, w) \text{ where } x = g(w)$$

We can write this as

$$y = f(g(w), w)$$

The three variables y, x and w are related to each other as shown in the figure below (referred to as a channel map).



It should be clear that w can now affect y through two channels – through its *direct* impact on y, and *indirectly* through its effect on x. So, we're really interested in knowing how a change in w will affect y, once we account for the direct and indirect effects. Because w has both a direct and indirect effect, it is the ultimate source of change in this model.

Note: A partial derivative (obtained using partial differentiation) is adequate for explaining the direct effect. However, when we have both direct and indirect effects, we need a total derivative.

Step 1: To find the total derivative, first find the total differential:

$$\begin{aligned} dy &= \frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial w}dw \\ &= f_x dx + f_w dw \end{aligned}$$

Step 2: Divide by the relevant differential. Because w is the driving force of change in this model, we want to find $\frac{dy}{dw}$. To do this, simply divide the total differential by dw :

$$\begin{aligned}\frac{dy}{dw} &= \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w}\frac{dw}{dw} \\ \frac{dy}{dw} &= \underbrace{\frac{\partial y}{\partial x}\frac{dx}{dw}}_{\text{indirect effect of } w} + \underbrace{\frac{\partial y}{\partial w}}_{\text{direct effect of } w}\end{aligned}$$

Be careful not to get your partial derivatives $\left(\frac{\partial y}{\partial w}\right)$ **mixed up with total derivatives** $\left(\frac{dy}{dw}\right)$**!**

Example 61 *Find* $\frac{dy}{dw}$ *given* $y = f(x, w) = 3x - w^2$ *where* $x = g(w) = 2w^2 + w + 4.$

First find the total differential:

$$dy = \frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial w}dw$$

Then find total derivative: (Because w is the ultimate source of change, we are interested in finding $\frac{dy}{dw}$*)*

$$\begin{aligned}\frac{dy}{dw} &= \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w}\frac{dw}{dw} \\ &= \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w}\end{aligned}$$

Now simply fill in the pieces:

$$\begin{aligned}\frac{dy}{dw} &= \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w} \\ &= (3)(4w + 1) + (-2w) \\ &= 10w + 3\end{aligned}$$

Example 62 *Find* $\frac{dy}{dw}$ *given* $y = f(x, w) = 4x^2 - 2w$ *where* $x = g(w) = w^2 + w - 3$.

First find the total differential:

$$dy = \frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial w}dw$$

Then find total derivative: (Because w is the ultimate source of change, we are interested in finding $\frac{dy}{dw}$*)*

$$\begin{aligned} \frac{dy}{dw} &= \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w}\frac{dw}{dw} \\ &= \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w} \\ &= (8x)(2w+1) + (-2) \\ &= 16wx + 8x - 2 \\ &= 16w\left(w^2 + w - 3\right) + 8\left(w^2 + w - 3\right) - 2 \\ &= 16w^3 + 24w^2 - 40w - 26 \end{aligned}$$

12.1 Economic Applications

Example 63 *Consider the utility function*

$$U = U(c, s)$$

where c is coffee and s is sugar. If $s = s(c)$, *we can re-write our utility function as*

$$U = U(c, s(c))$$

Because c is the driving force of change, we want to find the total derivative $\frac{dU}{dc}$.

First find the total differential:

$$dU = \frac{\partial U}{\partial c}dc + \frac{\partial U}{\partial s}ds$$

Then find total derivative: (Because c is the ultimate source of change, we are interested in finding $\frac{dU}{dc}$*)*

$$\begin{aligned} \frac{dU}{dc} &= \frac{\partial U}{\partial c}\frac{dc}{dc} + \frac{\partial U}{\partial s}\frac{ds}{dc} \\ &= \frac{\partial U}{\partial c} + \frac{\partial U}{\partial s}\frac{ds}{dc} \\ &= \frac{\partial U}{\partial c} + \frac{\partial U}{\partial s}s'(c) \end{aligned}$$

Example 64 *Let the production function be*

$$Q = Q(K, L, t) \text{ where } K = K(t) \text{ and } L = L(t)$$

The inclusion of t*, to denote time, reflects that the production function can change over time in response to technological change. In other words, we are dealing with a dynamic production function as opposed to a static production function.*

We can re-write this production function as

$$Q = Q(K(t), L(t), t)$$

The rate of change of output with respect to time is given by the total derivative $\frac{dQ}{dt}$*. First find the total differential:*

$$dQ = \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL + \frac{\partial Q}{\partial t} dt$$

Then find total derivative (with respect to time, which is the driving force of change):

$$\begin{aligned} \frac{dQ}{dt} &= \frac{\partial Q}{\partial K}\frac{dK}{dt} + \frac{\partial Q}{\partial L}\frac{dL}{dt} + \frac{\partial Q}{\partial t}\frac{dt}{dt} \\ &= Q_K K'(t) + Q_L L'(t) + Q_t \end{aligned}$$

13 Derivatives of Implicit Functions

The concept of total differentials enables us to find the derivatives of implicit functions.

13.1 Implicit Functions

A function given in the form of $y = f(x)$, for example

$$y = f(x) = 2x^2 \tag{1}$$

is called an *explicit function*, because the variable y is explicitly expressed as a function of x.

However, if the function is written in the equivalent form

$$y - 2x^2 = 0 \tag{2}$$

then we no longer have an explicit function. Rather, the function (1) is *implicitly* defined by the equation (2). When we are given an equation in the form of (2), therefore, the function $y = f(x)$ which it implies, and whose specific form may not even be known to us, is referred to as an *implicit function*.

In general, an equation of the form

$$F(y, x_1, \ldots, x_n) = 0$$

MAY also define an implicit function

$$y = f(x_1, \ldots, x_n)$$

The word MAY is important here. While it is always possible to transform an explicit function $y = f(x)$ into an equation $F(y, x) = 0$, the converse need not hold true. In other words, it is not necessarily the case that an equation of the form $F(y, x) = 0$ implicitly defines a function $y = f(x)$.

Example 65 *Consider the equation*

$$F(y, x) = x^2 + y^2 - 9 = 0$$

implies not a function, but a relation, because this equation describes a circle so that no unique value of y corresponds to each value of x.

Note, however, that if we restrict to nonnegative values (i.e. $y \geq 0$) then we will have the upper half of the circle only and that constitutes a function

$$y = +\sqrt{9 - x^2}$$

Similarly the lower half of the circle, where $y \leq 0$, constitutes another function

$$y = -\sqrt{9 - x^2}$$

But neither the left half nor the right half of the circle can qualify as a function.

The implicit function theorem provides us with the general conditions under which we can be sure that a given equation of the form

$$F(y, x_1, \ldots, x_n) = 0$$

does indeed define an implicit function

$$y = f(x_1, \ldots, x_n)$$

Theorem 1 *Given an equation of the form*

$$F(y, x_1, \ldots, x_n) = 0 \tag{3}$$

if

1. *F has continuous partial derivatives $F_y, F_1, \ldots, F_n$, and if*

2. *at a point $(y_0, x_{10}, \ldots, x_{n0})$ satisfying equation (3),$F_y \neq 0$*

then there exists an n-dimensional neighbourhood of $(y_0, x_{10}, \ldots, x_{n0})$ in which y is an implicitly defined function of the variables $x_1, \ldots, x_n$ in the form of $y = f(x_1, \ldots, x_n)$.

This implicit function f

(a) *gives (3) the status of an* identity *in the neighbourhood of $(y_0, x_{10}, \ldots, x_{n0})$, i.e. $F(y, x_1, \ldots, x_n) \equiv 0$.*
(b) *is continuous.*
(c) *has continuous partial derivatives $f_1, \ldots, f_n$.*

It is important to note that the conditions for the implicit function theorem are sufficient, but not necessary, conditions. Therefore it could be possible to find a point at which $F_y = 0$ but an implicit function may still exist around this point.

Example 66 *Suppose we have the equation*

$$F(y, x) = x^2 + y^2 - 9 = 0$$

We want to know whether it defines an implicit function.

1. *Does F have continuous partial derivatives?*

   Here the answer is yes.

$$\begin{aligned} F_y &= 2y \\ F_x &= 2x \end{aligned}$$

2. *For the points that satisfy the equation $F(y, x) = x^2 + y^2 - 9 = 0$, is $F_y \neq 0$?*

Since there are a whole range of possible (x, y) combinations that could satisfy the equation $F(y, x) = x^2 + y^2 - 9 = 0$, it might take some time to figure out whether $F_y \neq 0$ for each possible combination. So, take the opposite approach and see if you can calculate the values for which $F_y \neq 0$. Once you know the combinations of (x, y) for which $F_y \neq 0$, then you just need to check whether this falls in the range of possible (x, y) combinations which satisfy $F(y, x) = x^2 + y^2 - 9 = 0$.

So, $F_y = 2y$. Clearly, this will equal zero when y is zero.

When $y = 0, x$ values of -3 or 3 will satisfy $F(y, x) = x^2 + y^2 - 9 = 0$.

So, for the points $(-3, 0)$ and $(3, 0)$, $F_y = 0$.

But for all other combinations of (x, y), $F_y \neq 0$.

So for all possible combinations of (x, y) that satisfy $F(y, x) = x^2 + y^2 - 9 = 0$ except the two points $(-3, 0)$ and $(3, 0)$, $F_y \neq 0$ and therefore we will be able to find a neighbourhood of points for which the implicit function $y = f(x)$ is defined. Furthermore, given that the implicit function will be defined in this neighbourhood, we know that this function will be continuous, and will have continuous partial derivatives.

Graphically, this means that it is possible to draw, say, a rectangle around any point on the circle - except $(-3, 0)$ and $(3, 0)$ - such that the portion of the circle enclosed therein will constitute the graph of a function, with a unique y value for each value of x in that rectangle.

13.2 Derivatives of Implicit Functions

If you are given an equation of the form $F(y, x_1, \ldots, x_n) = 0$ and it is possible for you to re-write it as $y = f(x_1, \ldots, x_n)$, then you should go ahead and do this. Then, you can find the derivative as you have before.

Example 67 *The equation $F(y, x) = x^2 + y^2 - 9 = 0$ can easily be solved to give two separate functions:*

$$\begin{aligned} y^+ &= +\sqrt{9 - x^2} && \text{(upper half of circle)} \\ y^- &= -\sqrt{9 - x^2} && \text{(lower half of circle)} \end{aligned}$$

You can find the derivatives using the rules:

$$\begin{aligned} \frac{dy^+}{dx} &= \frac{1}{2}\left(9 - x^2\right)^{-1/2}(-2x) = \frac{-x}{y^+} && \left(y^+ \neq 0\right) \\ \frac{dy^-}{dx} &= -\frac{1}{2}\left(9 - x^2\right)^{-1/2}(-2x) = \frac{x}{y^-} && \left(y^- \neq 0\right) \end{aligned}$$

But what about cases where it's not so easy to re-write the equation $F(y, x_1, \ldots, x_n) = 0$ in terms of y? In this case, we make use of the implicit function rule:

If $F(y, x_1, \ldots, x_n) = 0$ defines an implicit function, then from the implicit function theorem, it follows that: (Refer back to the theorem for reassurance if you're feeling doubtful...)

$$F(y, x_1, \ldots, x_n) \equiv 0$$

This says that the LHS is identically equal to the RHS. If two expressions are identically equal, then their respective differentials must also be equal. (Consider this example: $a \equiv a$ if , then $da \equiv da$)

Thus,

$$dF(y, x_1, \ldots, x_n) \equiv d0 \quad \text{(we've just taken the differential of both sides)}$$

Now, write out the expression for the total differentials, dF, and $d0$.

$$F_y dy + F_1 dx_1 + F_2 dx_2 + \ldots + F_n dx_n = 0 \tag{4}$$

Now, the implicit function $y = f(x_1, \ldots, x_n)$ has the total differential

$$dy = f_1 dx_1 + f_2 dx_2 + \ldots + f_n dx_n$$

We can substitute this dy expression into (4) to get

$$(F_y f_1 + F_1) dx_1 + (F_y f_2 + F_2) dx_2 + \ldots + (F_y f_n + F_n) dx_n = 0$$

Since all the dx_i can vary independently from one another, for this equation to hold, each each bracket must individually vanish, i.e.

$$F_y f_i + F_i = 0 \qquad \text{(for all } i\text{)}$$

We divide through by F_y and solve for f_i:

$$f_i \equiv \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y} \qquad (i = 1, 2, \ldots, n)$$

In the simple case where $F(y, x) = 0$, the rule gives:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

To recap, the implicit function rule tells us that given an equation of the form $F(y, x_1, \ldots, x_n) = 0$, if an implicit function is defined, then its partial derivatives can be found using the formula:

$$f_i \equiv \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y} \qquad (i = 1, 2, \ldots, n)$$

This is a nice result because it means that even if you don't know what the implicit function looks like, you can still find its derivatives.

Example 68 *Suppose the equation* $F(y, x) = y - 3x^4 = 0$ *implicitly defines a function* $y = f(x)$*, then*

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\left(-12x^3\right)}{1} = 12x^3$$

Example 69 *Consider the equation of the circle* $F(y,x) = x^2 + y^2 - 9 = 0$. *Using the implicit function rule gives*

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{2y} = -\frac{x}{y}$$

Note that if $y = 0$, *then this expression is undefined BUT recall, that for this equation, when* $y = 0$ *the implicit function is not defined – see earlier example to re-check.*

You should also check that if you substitute in the functions y^+ *and* y^-, *you get the derivatives we found earlier using the rules.*

Example 70 *Suppose the equation* $F(y,x,w) = y^4 - 3x^4y^2 + 2wx - 1 = 0$ *implicitly defines a function* $y = f(x,w)$, *then*

$$\begin{aligned} \frac{\partial y}{\partial x} &= -\frac{F_x}{F_y} = -\frac{\left(-12x^3y^2 + 2w\right)}{(4y^3 - 6x^4y)} \\ \frac{\partial y}{\partial w} &= -\frac{F_w}{F_y} = -\frac{(2x)}{(4y^3 - 6x^4y)} \end{aligned}$$

Example 71 *Suppose the equation* $F(y,x,w) = xy^2 - 2xwy + 10wx + 5 = 0$ *implicitly defines a function* $y = f(x,w)$, *then*

$$\begin{aligned} \frac{\partial y}{\partial x} &= -\frac{F_x}{F_y} = -\frac{y^2 - 2wy + 10w}{2xy - 2xw} \\ \frac{\partial y}{\partial w} &= -\frac{F_w}{F_y} = -\frac{-2xy + 10x}{2xy - 2xw} \end{aligned}$$

Example 72 *Find* $\frac{\partial y}{\partial x}$ *and* $\frac{\partial y}{\partial w}$ *for any implicit function(s) that may be defined by the equation* $F(y,x,w) = y^3x^2 + w^3 + yxw - 3 = 0$.

First, we need to use the implicit function theorem to figure out whether an implicit function $y = f(x,w)$ *is defined or not.*

1. *Does* F *have continuous partial derivatives?*

$$\begin{aligned} F_y &= 3y^2x^2 + xw \\ F_x &= 2y^3x + yw \\ F_w &= 3w^2 + yx \end{aligned}$$

   Yes it does.

2. *Is* $F_y \neq 0$ *for the set of points that satisfy* $F(y,x,w) = y^3x^2 + w^3 + yxw - 3 = 0$?

   Well, one possible set of points (y,x,w) *that satisfies* $F(y,x,w) = 0$ *is* $(1,1,1)$. *At this point,* $F_y = 4$. *So, the second condition is met. This means that an implicit function is defined, at least in and around the neighbourhood of the point* $(1,1,1)$. *(We could show other points too, but one is enough)*

So, since an implicit function is defined (at least for some *neighbourhood of points), we can use the implicit function rule:*

$$\begin{aligned}\frac{\partial y}{\partial x} &= -\frac{F_x}{F_y} = -\frac{2y^3x + yw}{3y^2x^2 + xw} \\ \frac{\partial y}{\partial w} &= -\frac{F_w}{F_y} = -\frac{3w^2 + yx}{3y^2x^2 + xw}\end{aligned}$$

Example 73 *Find* $\frac{\partial y}{\partial x}$ *and* $\frac{\partial y}{\partial w}$ *for any implicit function(s) that may be defined by the equation* $F(y, x, w) = 3y^2x + x^2yw + yxw^2 - 16 = 0$.

First, we need to use the implicit function theorem to figure out whether an implicit function $y = f(x, w)$ *is defined or not.*

1. *Does* F *have continuous partial derivatives?*

$$\begin{aligned}F_y &= 6yx + x^2w + xw^2 \\ F_x &= 3y^2 + 2xyw + yw^2 \\ F_w &= x^2y + 2yxw\end{aligned}$$

   Yes it does.

2. *Is* $F_y \neq 0$ *for the set of points that satisfy* $F(y, x, w) = 3y^2x + x^2yw + yxw^2 - 16 = 0$?

   Well, one possible set of points (y, x, w) *that satisfies* $F(y, x, w) = 0$ *is* $(2, 1, 1)$. *At this point,* $F_y = 14$. *So, the second condition is met. This means that an implicit function is defined, at least in and around the neighbourhood of the point* $(2, 1, 1)$. *(We could show other points too, but one is enough)*

Now, we can use the implicit function rule:

$$\begin{aligned}\frac{\partial y}{\partial x} &= -\frac{F_x}{F_y} = -\frac{3y^2 + 2xyw + yw^2}{6yx + x^2w + xw^2} \\ \frac{\partial y}{\partial w} &= -\frac{F_w}{F_y} = -\frac{x^2y + 2yxw}{6yx + x^2w + xw^2}\end{aligned}$$

Example 74 *Assume that the equation* $F(Q, K, L) = 0$ *implicitly defines a production function* $Q = f(K, L)$, *then we can use the implicit function rule to find*

$$\begin{aligned}\frac{\partial Q}{\partial K} &= -\frac{F_K}{F_Q} \qquad \text{This is the marginal physical product of capital} \\ \frac{\partial Q}{\partial L} &= -\frac{F_L}{F_Q} \qquad \text{This is the marginal physical product of labour}\end{aligned}$$

BUT there's one more derivative we can find too:

$$\frac{\partial K}{\partial L} = -\frac{F_L}{F_K}$$

What is the meaning of $\frac{\partial K}{\partial L}$*? The partial sign implies that the other variable* Q *is being held constant, and so it simply gives us a description of the way in which capital inputs will change in response to a change in labour inputs in such a way as to keep output constant. Recall from production theory, that production is constant along an isoquant. (In the same way that utility is constant along an indifference curve). In other words,* $\frac{\partial K}{\partial L}$ *tells us something about moving along an isoquant (you must move along the isoquant if both* K *and* L *are changing). More precisely, it provides information about the slope of an isoquant. (which is usually negative). The absolute value of* $\frac{\partial K}{\partial L}$ *tells us the marginal rate of technical substitution between the two inputs, capital and labour.*

13.3 Application to the Simultaneous Equation Case

A generalised version of the implicit function theorem deals with the conditions under which a set of simultaneous equations

$$\begin{aligned} F^1(y_1, \ldots, y_m; x_1, \ldots, x_n) &= 0 \\ F^2(y_1, \ldots, y_m; x_1, \ldots, x_n) &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ F^m(y_1, \ldots, y_m; x_1, \ldots, x_n) &= 0 \end{aligned} \tag{5}$$

will assuredly define a set of implicit functions

$$\begin{aligned} y_1 &= f^1(x_1, \ldots, x_n) \\ y_2 &= f^2(x_1, \ldots, x_n) \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ y_m &= f^m(x_1, \ldots, x_n) \end{aligned} \tag{6}$$

The generalised version of the theorem states that:

Theorem 2 *Given the equation system* (5)*, if*

(a) the functions $F^1, \ldots, F^m$ *all have continuous partial derivatives with respect to all the* y *and* x *variables, and if*

(b) at a point $(y_{10},\ldots,y_{m0};x_{10},\ldots,x_{n0})$ *satisfying* (5)*, the following Jacobian determinant is non-zero:*

$$|J| \equiv \left|\frac{\partial\left(F^1,\ldots,F^m\right)}{\partial\left(y_1,\ldots,y_m\right)}\right| \equiv \begin{vmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_m} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F^m}{\partial y_1} & \frac{\partial F^m}{\partial y_2} & \cdots & \frac{\partial F^m}{\partial y_m} \end{vmatrix}$$

then there exists an n-dimensional neighbourhood of $(x_{10},\ldots,x_{n0})$ *in which the variables* $y_1,\ldots,y_m$ *are functions of the variables* $x_1,\ldots,x_n$ *in the form of* (6) *and*

1. *The implicit functions give* (5) *the status of identities in the neighbourhood around* $(y_{10},\ldots,y_{m0};x_{10},\ldots,x_{n0})$*.*
2. *The implicit functions* $f^1,\ldots,f^m$ *are continuous and have continuous partial derivatives with respect to all the* x *variables.*

As in the single equation case, it is possible to find the partial derivatives of the implicit functions directly from the m equations in (eqrefe7, without having to solve them for the y variables.

Since the equations in (5) have the status of identities in the neighbourhood around $(y_{10},\ldots,y_{m0};x_{10},\ldots,x_{n0})$, we can take the total differential of each of these

$$dF^j \equiv d0 \qquad (j=1,2,\ldots,m)$$

We can write out the expressions for dF^j and $d0$ and take the dx_i terms to the RHS to get

$$\begin{aligned} \frac{\partial F^1}{\partial y_1}dy_1+\frac{\partial F^1}{\partial y_2}dy_2+\ldots+\frac{\partial F^1}{\partial y_m}dy_m &= -\left(\frac{\partial F^1}{\partial x_1}dx_1+\frac{\partial F^1}{\partial x_2}dx_2+\ldots+\frac{\partial F^1}{\partial x_n}dx_n\right) \\ \frac{\partial F^2}{\partial y_1}dy_1+\frac{\partial F^2}{\partial y_2}dy_2+\ldots+\frac{\partial F^2}{\partial y_m}dy_m &= -\left(\frac{\partial F^2}{\partial x_1}dx_1+\frac{\partial F^2}{\partial x_2}dx_2+\ldots+\frac{\partial F^2}{\partial x_n}dx_n\right) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \frac{\partial F^m}{\partial y_1}dy_1+\frac{\partial F^m}{\partial y_2}dy_2+\ldots+\frac{\partial F^m}{\partial y_m}dy_m &= -\left(\frac{\partial F^m}{\partial x_1}dx_1+\frac{\partial F^m}{\partial x_2}dx_2+\ldots+\frac{\partial F^m}{\partial x_n}dx_n\right) \end{aligned} \tag{7}$$

Moreover, from (6), we can write the differentials of the y_j variables as

$$\begin{aligned} dy_1 &= \frac{\partial y_1}{\partial x_1}dx_1+\frac{\partial y_1}{\partial x_2}dx_2+\ldots+\frac{\partial y_1}{\partial x_n}dx_n \\ dy_2 &= \frac{\partial y_2}{\partial x_1}dx_1+\frac{\partial y_2}{\partial x_2}dx_2+\ldots+\frac{\partial y_2}{\partial x_n}dx_n \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ dy_m &= \frac{\partial y_m}{\partial x_1}dx_1+\frac{\partial y_m}{\partial x_2}dx_2+\ldots+\frac{\partial y_m}{\partial x_n}dx_n \end{aligned} \tag{8}$$

and these can be used to eliminate the dy_j expressions in (7). But this would be very messy, so let's simplify matters by considering only what would happen when x_1 alone changes while all the other variables $x_2, \ldots, x_n$ remain constant.

Letting $dx_1 \neq 0$, but setting $dx_2 = \ldots = dx_n = 0$ in (7) and (8), then substituting (8) into (7) and dividing through by $dx_1 \neq 0$, we obtain the equation system

$$\begin{array}{rcl}
\frac{\partial F^1}{\partial y_1}\left(\frac{\partial y_1}{\partial x_1}\right) + \frac{\partial F^1}{\partial y_2}\left(\frac{\partial y_2}{\partial x_1}\right) + \ldots + \frac{\partial F^1}{\partial y_m}\left(\frac{\partial y_m}{\partial x_1}\right) & = & -\frac{\partial F^1}{\partial x_1} \\
\frac{\partial F^2}{\partial y_1}\left(\frac{\partial y_1}{\partial x_1}\right) + \frac{\partial F^2}{\partial y_2}\left(\frac{\partial y_2}{\partial x_1}\right) + \ldots + \frac{\partial F^2}{\partial y_m}\left(\frac{\partial y_m}{\partial x_1}\right) & = & -\frac{\partial F^2}{\partial x_1} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\
\frac{\partial F^m}{\partial y_1}\left(\frac{\partial y_1}{\partial x_1}\right) + \frac{\partial F^m}{\partial y_2}\left(\frac{\partial y_2}{\partial x_1}\right) + \ldots + \frac{\partial F^m}{\partial y_m}\left(\frac{\partial y_m}{\partial x_1}\right) & = & -\frac{\partial F^m}{\partial x_1}
\end{array} \tag{9}$$

This may look complicated, but notice that the expressions in brackets constitute the partial derivatives of the implicit functions with respect to x_1 that we want to find. They should therefore be regarded as the "variables" to be solved for in (9). The other derivatives are the partial derivatives of the F^j functions given in (5) and would all take specific values when evaluated at the point $(y_{10}, \ldots, y_{m0}; x_{10}, \ldots, x_{n0})$ - the point around which the implicit functions are defined - and so they can be treated as given constants.

These facts make (9) a linear equation system, and it can be written in matrix form as

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_m} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F^m}{\partial y_1} & \frac{\partial F^m}{\partial y_2} & \cdots & \frac{\partial F^m}{\partial y_m} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial y_1}{\partial x_1}\right) \\ \left(\frac{\partial y_2}{\partial x_1}\right) \\ \vdots \\ \left(\frac{\partial y_m}{\partial x_1}\right) \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial x_1} \\ -\frac{\partial F^2}{\partial x_1} \\ \vdots \\ -\frac{\partial F^m}{\partial x_1} \end{bmatrix} \tag{10}$$

Note that the coefficient matrix is just the Jacobian matrix J and the Jacobian determinant $|J|$ is known to be non-zero under the conditions of the implicit function theorem, there should be a unique solution to (10). By Cramer's rule, this solution can be expressed as

$$\left(\frac{\partial y_j}{\partial x_1}\right) = \frac{|J_j|}{|J|} \qquad (j = 1, 2, \ldots m)$$

By suitable adaptation of this procedure, the partial derivatives of the implicit functions with respect to the other variables $x_2, \ldots, x_n$ can also be found.

Example 75 *The following three equations*

$$\begin{aligned} F^1(x, y, w; z) &= xy - w = 0 \\ F^2(x, y, w; z) &= y - w^3 - 3z = 0 \\ F^3(x, y, w; z) &= w^3 + z^3 - 2zw = 0 \end{aligned}$$

are satisfied at the point $P : (x, y, w; z) = \left(\frac{1}{4}, 4, 1, 1\right)$.

The F^j *functions obviously possess continuous partial derivatives. Thus, if the Jacobian determinant* $|J| \neq 0$ *at point* P*, we can use the implicit function theorem to find* $\frac{\partial x}{\partial z}$.

First, we take the total differential of the system

$$\begin{aligned} ydx + xdy - dw &= 0 \\ dy - 3w^2dw - 3dz &= 0 \\ \left(3w^2 - 2z\right)dw + \left(3z^2 - 2w\right)dz &= 0 \end{aligned}$$

Moving the exogenous differential dz *to the RHS and writing in matrix form we get*

$$\begin{bmatrix} y & x & -1 \\ 0 & 1 & -3w^2 \\ 0 & 0 & 3w^2 - 2z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dw \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2w - 3z^2 \end{bmatrix} dz$$

where the coefficient matrix of the LHS is the Jacobian

$$|J| = \begin{vmatrix} F_x^1 & F_y^1 & F_w^1 \\ F_x^2 & F_y^2 & F_w^2 \\ F_x^3 & F_y^3 & F_w^3 \end{vmatrix} = \begin{vmatrix} y & x & -1 \\ 0 & 1 & -3w^2 \\ 0 & 0 & 3w^2 - 2z \end{vmatrix} = y\left(3w^2 - 2z\right)$$

At the point P*,* $|J| = 4 \neq 0$*. Therefore the implicit function rule applies and*

$$\begin{bmatrix} y & x & -1 \\ 0 & 1 & -3w^2 \\ 0 & 0 & 3w^2 - 2z \end{bmatrix} \begin{bmatrix} \left(\frac{\partial x}{\partial z}\right) \\ \left(\frac{\partial y}{\partial z}\right) \\ \left(\frac{\partial w}{\partial z}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2w - 3z^2 \end{bmatrix}$$

Use Cramer's rule to find an expression for $\frac{\partial x}{\partial z}$:

$$\begin{aligned}
\left(\frac{\partial x}{\partial z}\right) &= \frac{\begin{vmatrix} 0 & x & -1 \\ 3 & 1 & -3w^2 \\ 2w-3z^2 & 0 & 3w^2-2z \end{vmatrix}}{|J|} \\
&= \frac{\begin{vmatrix} 0 & \frac{1}{4} & -1 \\ 3 & 1 & -3 \\ -1 & 0 & 1 \end{vmatrix}}{4} \\
&= -\frac{1}{4}
\end{aligned}$$

13.4 Application to Market Model

Example 76 *The market for Marc Jacobs handbags is described by the following set of equations*

$$\begin{aligned}
Q_d &= Q_s \\
Q_d &= D(P, G) \\
Q_s &= S(P, N)
\end{aligned}$$

where G *is the price of substitutes and* N *is the price of inputs, and* G *and* N *are exogenously given. The following assumptions are imposed*

$$\begin{aligned}
\frac{\partial D}{\partial P} &< 0, \frac{\partial D}{\partial G} > 0 \\
\frac{\partial S}{\partial P} &> 0, \frac{\partial S}{\partial N} < 0
\end{aligned}$$

Use the implicit-function rule to find and sign the derivatives $\frac{\partial P^*}{\partial G}, \frac{\partial Q^*}{\partial G}, \frac{\partial P^*}{\partial N}$ *and* $\frac{\partial Q^*}{\partial N}$.

First, express the market model as a two-equation system by letting $Q = Q_d = Q_s$*:*

$$\begin{aligned}
Q &= D(P, G) \\
Q &= S(P, N)
\end{aligned}$$

or equivalently:

$$\begin{aligned}
F^1(P, Q; G, N) &= D(P, G) - Q = 0 \\
F^2(P, Q; G, N) &= S(P, N) - Q = 0
\end{aligned}$$

Next, check the conditions for the implicit function theorem:

1.

$$\begin{aligned}F_P^1 &= \frac{\partial D}{\partial P}\\ F_Q^1 &= -1\\ F_G^1 &= \frac{\partial D}{\partial G}\\ F_N^1 &= 0\end{aligned}$$

$$\begin{aligned}F_P^2 &= \frac{\partial S}{\partial P}\\ F_Q^2 &= -1\\ F_G^2 &= 0\\ F_N^2 &= \frac{\partial S}{\partial N}\end{aligned}$$

Therefore, continuous partial derivatives with respect to all endogenous and exogenous variables exist.

2.

$$\begin{aligned}|J| &= \begin{vmatrix}F_P^1 & F_Q^1\\ F_P^2 & F_Q^2\end{vmatrix}\\ &= \begin{vmatrix}\frac{\partial D}{\partial P} & -1\\ \frac{\partial S}{\partial P} & -1\end{vmatrix}\\ &= -\frac{\partial D}{\partial P}+\frac{\partial S}{\partial P}\\ &= \frac{\partial S}{\partial P}-\frac{\partial D}{\partial P}\\ &> 0\end{aligned}$$

Therefore, $|J| \neq 0$

Conditions for implicit function satisfied, and so system implicitly defines the functions $P^*(G,N)$ *and* $Q^*(G,N)$.

Now, we can use the implicit function rule:

First, take the total differential of each equation:

$$\begin{aligned}dF^1 = F_P^1 dp + F_Q^1 dQ + F_G^1 dG + F_N^1 dN &= 0\\ \Rightarrow \frac{\partial D}{\partial P}dp - 1dQ + \frac{\partial D}{\partial G}dG + 0 &= 0\\ \frac{\partial D}{\partial P}dp - 1dQ &= -\frac{\partial D}{\partial G}dG \qquad (11)\end{aligned}$$

$$\begin{aligned} dF^2 = F_P^2 dp + F_Q^2 dQ + F_G^2 dG + F_N^2 dN &= 0 \\ \Rightarrow \frac{\partial S}{\partial P}dp - 1dQ + 0 + \frac{\partial S}{\partial N}dN &= 0 \\ \frac{\partial S}{\partial P}dP - 1dQ &= -\frac{\partial S}{\partial N}dN \end{aligned} \tag{12}$$

Putting equations (11) *and* (12) *in matrix form*

$$\begin{bmatrix} \frac{\partial D}{\partial P} & -1 \\ \frac{\partial S}{\partial P} & -1 \end{bmatrix} \begin{bmatrix} dP \\ dQ \end{bmatrix} = \begin{bmatrix} -\frac{\partial D}{\partial G} \\ 0 \end{bmatrix} dG + \begin{bmatrix} 0 \\ -\frac{\partial S}{\partial N} \end{bmatrix} dN \tag{13}$$

Note that the coefficient matrix is the Jacobian matrix J.

To find $\frac{\partial P^*}{\partial G}$ *and* $\frac{\partial Q^*}{\partial G}$ *we partially differentiate with respect with* G*, holding* N *constant which implies that* $dN = 0$*. Setting* $dN = 0$ *and dividing through by* dG *in* (13) *gives:*

$$\begin{bmatrix} \frac{\partial D}{\partial P} & -1 \\ \frac{\partial S}{\partial P} & -1 \end{bmatrix} \begin{bmatrix} \frac{\partial P^*}{\partial G} \\ \frac{\partial Q^*}{\partial G} \end{bmatrix} = \begin{bmatrix} -\frac{\partial D}{\partial G} \\ 0 \end{bmatrix}$$

(Note the partial derivative *signs - we are differentiating with respect to G, holding N constant).*

Use Cramer's rule to solve for $\frac{\partial P^*}{\partial G}$ *and* $\frac{\partial Q^*}{\partial G}$ *:*

$$\begin{aligned} \frac{\partial P^*}{\partial G} &= \frac{\begin{vmatrix} -\frac{\partial D}{\partial G} & -1 \\ 0 & -1 \end{vmatrix}}{|J|} \\ &= \frac{\frac{\partial D}{\partial G}}{\frac{\partial S}{\partial P} - \frac{\partial D}{\partial P}} \\ &> 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial Q^*}{\partial G} &= \frac{\begin{vmatrix} \frac{\partial D}{\partial P} & -\frac{\partial D}{\partial G} \\ \frac{\partial S}{\partial P} & 0 \end{vmatrix}}{|J|} \\ &= \frac{\frac{\partial D}{\partial G}\frac{\partial S}{\partial P}}{\frac{\partial S}{\partial P} - \frac{\partial D}{\partial P}} \\ &> 0 \end{aligned}$$

To find $\dfrac{\partial P^*}{\partial N}$ *and* $\dfrac{\partial Q^*}{\partial N}$ *we partially differentiate with respect with* N*, holding* G *constant which implies that* $dG = 0$*. Setting* $dG = 0$ *and dividing through by* dN *in* (13) *gives:*

$$\begin{bmatrix} \frac{\partial D}{\partial P} & -1 \\ \frac{\partial S}{\partial P} & -1 \end{bmatrix} \begin{bmatrix} \dfrac{\partial P^*}{\partial N} \\ \dfrac{\partial Q^*}{\partial N} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\partial S}{\partial N} \end{bmatrix}$$

(Note the partial derivative *signs - we are differentiating with respect to* N*, holding* G *constant).*

Use Cramer's rule to solve for $\dfrac{\partial P^*}{\partial N}$ *and* $\dfrac{\partial Q^*}{\partial N}$:

$$\begin{aligned} \frac{\partial P^*}{\partial N} &= \frac{\begin{vmatrix} 0 & -1 \\ -\frac{\partial S}{\partial N} & -1 \end{vmatrix}}{|J|} \\ &= \frac{-\frac{\partial S}{\partial N}}{\frac{\partial S}{\partial P} - \frac{\partial D}{\partial P}} \\ &> 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial Q^*}{\partial N} &= \frac{\begin{vmatrix} \frac{\partial D}{\partial P} & 0 \\ \frac{\partial S}{\partial P} & -\frac{\partial S}{\partial N} \end{vmatrix}}{|J|} \\ &= \frac{-\frac{\partial D}{\partial P}\frac{\partial S}{\partial N}}{\frac{\partial S}{\partial P} - \frac{\partial D}{\partial P}} \\ &< 0 \end{aligned}$$

References

[1] Chiang, A.C. and Wainwright, K. 2005. *Fundamental Methods of Mathematical Economics*, 4th ed. McGraw-Hill International Edition.

[2] Pemberton, M. and Rau, N.R. 2001. *Mathematics for Economists: An introductory textbook*, Manchester: Manchester University Press.