Mathematics for Economists

Linear Algebra

Section 1: Linear Algebra

ECO4112F 2011

Linear (matrix) algebra is a very useful tool in mathematical modelling as it allows us to deal with (among other things) large systems of equations, with relative ease. As the name implies however, linear algebra applies only to linear equations - i.e., equations with only a first order polynomial.

An understanding of matrix algebra is important for most facets of economic theory. Solving mathematical models that are based on a large number of simultaneous equations (such as the national income model, or market models) requires the use of matrix algebra. Once we get past two linear equations with more than two endogenous variables, it becomes tedious and time consuming to solve by simple substitution. Matrix algebra provides an easy method to solve these systems of equations. Similarly, matrices are important for econometrics. Any data set we use can and should be thought of as a large matrix of numerical elements. The core essence of econometrics is to solve for the coefficient values attached to each numerical element in the data set.

1 Vectors

Definition 1 *A vector is a list of numbers.*

There are *row* vectors, e.g. $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and *column* vectors, e.g. $\mathbf{c} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

The number in the i^{th} position of the vector $\mathbf{a}$ is denoted by a_i and is referred to as the i^{th} *component* of $\mathbf{a}$. So $r_2 = 2$ and $c_3 = 6$.

A vector with n components is called an n-vector.

The order of the components of a vector matters. Thus, if $\mathbf{x}$ and $\mathbf{y}$ are both n-vectors,

$$\mathbf{x} = \mathbf{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, \ldots, n$$

1.1 Vector arithmetic

Addition of two vectors $\mathbf{a}$ and $\mathbf{b}$ can only be performed if they have the same number of components and is performed component by component:

Example 1

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 7 \end{bmatrix}$$

Multiplication by a scalar is also performed component by component:

Example 2

$$(-2)\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ -4 \end{bmatrix}$$

1.1.1 Laws

(a) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (Commutative law of vector addition)

(b) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (Associative law of vector addition)

(c) $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$

(d) $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$

(e) $\lambda(\mu\mathbf{a}) = \mu(\lambda\mathbf{a}) = (\lambda\mu)\mathbf{a}$

1.2 Linear Dependence

A *linear combination* of two vectors **a** and **b** is a vector of the form

$$\alpha\mathbf{a} + \beta\mathbf{b}$$

where α and β are scalars.

Suppose we have a set of k n-vectors, $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^k$.

Definition 2 *The vectors* $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^k$ *are* linearly dependent *if it is possible to express one of them as a linear combination of the others.*

The vectors $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^k$ *are* linearly independent *if none of the vectors can be expressed as a linear combination of the others.*

Criterion 1 *A simple criterion for linear dependence is:*

$\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^k$ *are linearly dependent if and only if there exist scalars* $\alpha_1, \alpha_2, \ldots, \alpha_k$, not all zero*, such that*

$$\alpha_1\mathbf{b}^1 + \alpha_2\mathbf{b}^2 + \ldots + \alpha_k\mathbf{b}^k = \mathbf{0} \tag{1}$$

Thus, to test whether a given set of vectors are linearly dependent or independent, start with (1). If you can show that the only scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ which satisfy this are all zero, then the vectors are linearly independent; if you can find $\alpha_1, \alpha_2, \ldots, \alpha_k$ that satisfy (1) and are *not* all zero, then the vectors are linearly dependent.

Example 3 *Consider the vectors*

$$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Are $\mathbf{a}$ *and* $\mathbf{b}$ *linearly independent?*
Are $\mathbf{a}$*,* $\mathbf{b}$ *and* $\mathbf{c}$ *linearly independent?*
$\mathbf{a}$ *and* $\mathbf{b}$ *are linearly independent, since the only scalars* α *and* β *for which*

$$\begin{aligned} 2\alpha + \beta &= 0 \\ \alpha + 2\beta &= 0 \end{aligned}$$

are given by $\alpha = \beta = 0$*.*
$\mathbf{a}$*,* $\mathbf{b}$ *and* $\mathbf{c}$ *linearly dependent, since*

$$\mathbf{a} + \mathbf{b} - 3\mathbf{c} = \mathbf{0}$$

2 Matrices

Definition 3 *A matrix is a rectangular array of numbers, e.g.* $\mathbf{A} = \begin{bmatrix} 9 & 5 & 7 \\ 4 & 8 & 6 \end{bmatrix}$.

The number of rows and number of columns in a matrix together define the *dimension* of the matrix. A matrix with m rows and n columns is called an $m \times n$ matrix.

The number in the i^{th} row and j^{th} column of a matrix $\mathbf{A}$ is called the (i, j) entry of $\mathbf{A}$: denoted a_{ij}.
So using our matrix $\mathbf{A}$ above, $a_{23} = 6$.

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are equal only if they are exactly the same, with the same number of rows, the same number of columns and the same entries in the same order.

Note that we can think of vectors as special cases of matrices: a row vector is simply a $1 \times n$ matrix, and a column vector is simply an $n \times 1$ matrix.

3 Matrix Operations

3.1 Addition and Subtraction of Matrices

This is analogous to the vector case. Two matrices $\mathbf{A}$ and $\mathbf{B}$ can be added if and only if they have the same number of rows and the same number of columns (i.e. the same dimension).

When this dimension requirement is met, the matrices are said to be *conformable for addition.* In that case, addition of $\mathbf{A}$ and $\mathbf{B}$ is performed entry by entry:

$$\mathbf{A}+\mathbf{B}=\mathbf{C} \quad \text{where } c_{ij}=a_{ij}+b_{ij}$$

The sum matrix $\mathbf{C}$ must have the same dimension as the component matrices $\mathbf{A}$ and $\mathbf{B}$.

We can define subtraction similarly for two conformable matrices $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A}-\mathbf{B}=\mathbf{D} \quad \text{where } d_{ij}=a_{ij}-b_{ij}$$

Alternatively, we can define subtraction by letting

$$-\mathbf{A}=(-1)\,\mathbf{A}, \qquad \mathbf{A}-\mathbf{B}=\mathbf{A}+(-\mathbf{B})$$

3.1.1 Laws of matrix addition

1. Matrix addition satisfies the commutative law:

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$$

2. Matrix addition satisfies the associative law:

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$$

Example 4 *Given*

$$\mathbf{A}=\begin{bmatrix}4 & 3 & 1\\ 2 & 6 & 5\\ 1 & 0 & 2\end{bmatrix} \quad \textit{and } \mathbf{B}=\begin{bmatrix}1 & 0 & 2\\ 3 & 2 & 1\\ 4 & 1 & 2\end{bmatrix}$$

$$\begin{aligned}\mathbf{A}+\mathbf{B} &= \begin{bmatrix}5 & 3 & 3\\ 5 & 8 & 6\\ 5 & 1 & 4\end{bmatrix}\\ \mathbf{A}-\mathbf{B} &= \begin{bmatrix}3 & 3 & -1\\ -1 & 4 & 4\\ -3 & -1 & 0\end{bmatrix}\end{aligned}$$

3.2 Scalar Multiplication

This is again analogous to the vector case. To multiply a matrix by a scalar, multiply every element in that matrix by the scalar.

Example 5

$$\mathbf{A} = \begin{bmatrix} 8 & 0 & 9 \\ 1 & -4 & 1 \end{bmatrix}$$

$$2\mathbf{A} = \begin{bmatrix} 16 & 0 & 18 \\ 2 & -8 & 2 \end{bmatrix}$$

3.3 Matrix Multiplication

Suppose we have two matrices $\mathbf{A}$ and $\mathbf{B}$ and we want to find the product $\mathbf{AB}$. The conformability condition for multiplication is that the column dimension of $\mathbf{A}$ must be equal to the row dimension of $\mathbf{B}$. When the product $\mathbf{AB}$ is defined, it has the same number of rows as $\mathbf{A}$ and the same number of columns as $\mathbf{B}$:

$$\underset{m\times n}{\mathbf{A}}\ \underset{n\times s}{\mathbf{B}} = \underset{m\times s}{\mathbf{C}}$$

Denote the rows of $\mathbf{A}$ by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ and the columns of $\mathbf{B}$ by $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^s$. Then $\mathbf{AB}$ is the $m \times s$ matrix whose (i, k) entry is the row-column product $\mathbf{a}_i\mathbf{b}^k$ for all relevant i and k. For instance to find the $(1, 1)$ entry of $\mathbf{AB}$ we should take the first row in $\mathbf{A}$ and the first column in $\mathbf{B}$ and then pair the elements together sequentially, multiply out each pair, and take the sum of the resulting products, to get

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{s1}$$

Example 6 *Given*

$$\mathbf{A} = \begin{bmatrix} 5 & 7 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 5 & 2 & 0 \\ 6 & 9 & 1 \end{bmatrix}$$

Find $\mathbf{AB}$.

First, we must check that the matrices are conformable: the column dimension of $\mathbf{A}$ *is* 2 *and the row dimension of* $\mathbf{B}$ *is* 2, *so the product* $\mathbf{AB} = \mathbf{C}$ *is defined.* $\mathbf{C}$ *will be a* 1×3 *matrix.*

Now $\mathbf{AB} = \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \end{bmatrix}$.

To find c_{11}, *we use row* 1 *of* $\mathbf{A}$ *and column* 1 *of* $\mathbf{B}$:

$$\begin{aligned} c_{11} &= (5 \times 5) + (7 \times 6) \\ &= 67 \end{aligned}$$

To find c_{12}, *we use row* 1 *of* $\mathbf{A}$ *and column* 2 *of* $\mathbf{B}$:

$$\begin{aligned} c_{11} &= (5 \times 2) + (7 \times 9) \\ &= 73 \end{aligned}$$

To find c_{13}*, we use row* 1 *of* **A** *and column* 3 *of* **B** :

$$\begin{aligned} c_{11} &= (5\times 0)+(7\times 1) \\ &= 7 \end{aligned}$$

Thus,

$$\mathbf{AB}=\mathbf{C}=\begin{bmatrix}67 & 73 & 7\end{bmatrix}$$

Example 7 *Given*

$$\mathbf{A}=\begin{bmatrix}1 & 5\\ 2 & -3\\ 4 & -8\end{bmatrix} \text{ and } \mathbf{B}=\begin{bmatrix}2 & -7 & 3 & 0\\ 1 & 0 & 4 & -5\end{bmatrix}$$

Find **AB** *and* **BA**.

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix}(1\times 2+5\times 1) & (-1\times 7+5\times 0) & (1\times 3+5\times 4) & (1\times 0-5\times 5)\\ (2\times 2-3\times 1) & (-2\times 7-3\times 0) & (2\times 3-3\times 4) & (2\times 0+3\times 5)\\ (4\times 2-8\times 1) & (-4\times 7-8\times 0) & (4\times 3-8\times 4) & (4\times 0+8\times 5)\end{bmatrix} \\ &= \begin{bmatrix}7 & -7 & 23 & -25\\ 1 & -14 & -6 & 15\\ 0 & -28 & -20 & 40\end{bmatrix} \end{aligned}$$

BA *is not defined because the matrices are not conformable for multiplication.*

3.3.1 Rules of matrix multiplication

1. The associative law of matrix multiplication:

$$(\mathbf{AB})\,\mathbf{C}=\mathbf{A}\,(\mathbf{BC})$$

2. The distributive laws of matrix multiplication:

$$\begin{aligned} (\mathbf{A}+\mathbf{B})\,\mathbf{C} &= \mathbf{AC}+\mathbf{BC} \\ \mathbf{A}\,(\mathbf{B}+\mathbf{C}) &= \mathbf{AB}+\mathbf{AC} \end{aligned}$$

3. Scalars:

$$(\lambda\mathbf{A})\,\mathbf{B}=\lambda\,(\mathbf{AB})=\mathbf{A}\,(\lambda\mathbf{B})$$

Note that *matrix multiplication is not commutative.* In general,

$$\mathbf{AB}\neq\mathbf{BA}$$

Because matrix multiplication is not commutative, the instruction "multiply **A** by **B**" is ambiguous and should be avoided Instead, use the terms premultiply or postmultiply: premultiplication of **A** by **B** means forming the product **BA**, postmultiplication of **A** by **B** means forming the product **AB**.

3.4 Summary of matrix arithmetic

Matrix arithmetic is just like ordinary arithmetic, except for the following:

1. Addition and multiplication are subject to dimension restrictions: $\mathbf{A}+\mathbf{B}$ is defined only when $\mathbf{A}$ and $\mathbf{B}$ have the same number of rows and the same number of columns; $\mathbf{AB}$ is defined only when the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$.

2. Matrix multiplication is not commutative.

3. There is no such thing as matrix division.

4 Systems of linear equations

We can express any system of *linear* equations in the form $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{A}$ is referred to as the coefficient matrix, and $\mathbf{x}$ as the vector of unknowns.

Example 8 *Rewrite the following system of linear equations using matrix algebra:*

$$\begin{aligned} 6x_1+3x_2+x_3 &= 22 \\ x_1+4x_2-2x_3 &= 12 \\ 4x_1-x_2+5x_3 &= 10 \end{aligned}$$

We can write this system as $\mathbf{Ax}=\mathbf{b}$ *where*

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \\ \mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \mathbf{b} &= \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} \end{aligned}$$

We refer to $\mathbf{A}$ *as the coefficient matrix, and* $\mathbf{x}$ *as the vector of unknowns.*

Example 9 *Rewrite the following national income model in matrix form:*

$$\begin{aligned} Y &= C+I+G \\ C &= a+b(Y-T) \qquad a>0, 0<b<1 \\ T &= tY \qquad 0<t<1 \end{aligned}$$

where Y is national income, C is (planned) consumption expenditure, I is investment expenditure, G is government expenditure and T is taxes.

We first need to re-arrange the equations so that the endogenous variables are on the LHS and the exogenous variables are on the RHS:

$$\begin{aligned} Y - C &= I + G \\ C - bY + bT &= a \\ T - tY &= 0 \end{aligned}$$

We can now rewrite this system in matrix form:

$$\begin{bmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} I+G \\ a \\ 0 \end{bmatrix}$$

5 Square matrices

Definition 4 *An $n \times n$ matrix is called a square matrix of order n.*

Example 10

$$\mathbf{A} = \begin{bmatrix} 5 & 6 & 2 \\ 4 & 2 & 1 \\ 8 & 9 & 2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 4 & -5 & 5 & 0 \\ 7 & 0 & 6 & 0 \\ 1 & 9 & 5 & 8 \end{bmatrix}$$

$\mathbf{A}$ *is a square matrix of order* 3.
$\mathbf{B}$ *is not a square matrix.*

5.1 Triangular and diagonal matrices

Given a 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

we call a_{11}, a_{22} and a_{33} the *diagonal* entries of $\mathbf{A}$, the other six entries are *off-diagonal.*

A square matrix is said to be:

- *upper triangular* if all entries below the diagonal are zero,
- *lower triangular* if all entries above the diagonal are zero.

An upper triangular matrix with no zeros on the diagonal is called a *regular upper triangular* matrix (RUT for short).

Definition 5 *A* triangular matrix *is a matrix which is either upper or lower triangular.*

Definition 6 *A* diagonal matrix *is one that is both upper and lower triangular.*

Example 11

$$\mathbf{A} = \begin{bmatrix} 7 & 10 & 9 \\ 0 & 1 & 18 \\ 0 & 0 & 2 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 3 & 0 \\ 4 & 2 & 5 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

- **A** *is upper triangular*
- **B** *is lower triangular*
- **C** *is a diagonal matrix.*

5.2 Trace

Definition 7 *The* trace *of a square matrix is the sum of its diagonal entries. For an* $n \times n$ *matrix* **A***,*

$$tr(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$

Example 12

$$\mathbf{A} = \begin{bmatrix} 4 & -3 & 4 \\ 6 & -1 & -3 \\ 0 & 3 & -4 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} -5 & 2 & -2 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} tr(\mathbf{A}) &= 4 - 1 - 4 = -1 \\ tr(\mathbf{B}) &= -5 + 3 + 2 = 0 \end{aligned}$$

6 Identity and Null Matrices

6.1 Identity Matrices

Definition 8 *An identity matrix is a* square *matrix with* $1s$ *on its diagonal and* $0s$ *everywhere else. It is denoted by the symbol* $\mathbf{I}$, *or* $\mathbf{I}_n$ *where the subscript* n *serves to indicate its row (as well as column) dimension.*

Thus,

$$\mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The identity matrix plays the same role in matrix multiplication that the number 1 plays in ordinary arithmetic: for any matrix $\mathbf{A}$,

$$\mathbf{IA} = \mathbf{A} = \mathbf{AI}$$

6.2 Null Matrices

Definition 9 *A null matrix - or zero matrix - is simply a matrix whose elements are all zero. It is denoted by* $\mathbf{0}$.

Unlike $\mathbf{I}$, the null matrix is not restricted to being square.

The zero matrix plays the same role for matrices that the number 0 does in ordinary arithmetic:

$$\mathbf{A} + \mathbf{0} = \mathbf{A} = \mathbf{0} + \mathbf{A}$$
$$\mathbf{A0} = \mathbf{0}$$
$$\mathbf{0A} = \mathbf{0}$$

7 Echelon matrices

Definition 10 *An echelon matrix is a matrix, not necessarily square, with the following two properties:*

1. *There is at least one non-zero entry; rows consisting entirely of zeros, if any, lie below rows with at least one non-zero entry.*
2. *In each non-zero row after the first, the left-most non-zero entry lies to the right of the left-most non-zero entry in the preceding row.*

These properties give us a 'staircase' pattern, where each $\bigstar$ denotes a non-zero number, and the entries marked $\bullet$ may be zero or non-zero:

$$\begin{bmatrix} \bigstar & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bigstar & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bigstar & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In each of the non-zero rows of an echelon matrix:

- the left-most non-zero entry is called the *pivot,*
- the columns containing the pivots are said to be *basic columns.*

For any echelon matrix

number of pivots = number of non-zero rows = number of basic columns.

Definition 11 *A row echelon matrix in which each pivot is a* 1 *and in which each column containing a pivot contains no other non-zero entries is said to be in* reduced row echelon *form.*

For instance,

$$\begin{bmatrix} 1 & 0 & \bullet & \bullet & 0 & \bullet \\ 0 & 1 & \bullet & \bullet & 0 & \bullet \\ 0 & 0 & 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.1 Echelon systems

We now explain how to solve a system of equations of the form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is an echelon matrix.

Our first example is the case where $\mathbf{A}$ is a RUT. The system is solved by back-substitution, giving a unique solution.

Example 13 *Let*

$$\mathbf{A} = \begin{bmatrix} 2 & 7 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

Then the system of linear equations $\mathbf{Ax} = \mathbf{b}$ *may be written*

$$\begin{aligned} 2x_1 + 7x_2 + x_3 &= 2 \\ 3x_2 - 2x_3 &= 7 \\ 4x_3 &= 4 \end{aligned}$$

Solve by back substitution to get the unique solution $x_1 = -10, x_2 = 3, x_3 = 1$.

Our second example considers the case where the basic columns of $\mathbf{A}$ form a RUT, and there are also one or more non-basic columns. To solve the system, we assign arbitrary values to non-basic unknowns, and then solve for the basic unknowns by back-substitution.

Example 14 *Solve the system* $\mathbf{Ax} = \mathbf{b}$, *where*

$$\mathbf{A} = \begin{bmatrix} 2 & -4 & 1 & -8 \\ 0 & 0 & 5 & 10 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ 20 \end{bmatrix}$$

The first and third columns are basic. We therefore assign arbitrary values λ *and* μ *to* x_2 *and* x_4, *giving the system*

$$\begin{aligned} 2x_1 + x_3 &= 4\lambda + 8\mu - 4 \\ 5x_3 &= -10\mu + 20 \end{aligned}$$

We now find x_1 *and* x_3 *by back-substitution:*

$$\begin{aligned} x_3 &= 4 - 2\mu \\ x_1 &= \frac{1}{2}(4\lambda + 8\mu - 4 + 2\mu - 4) = 2\lambda + 5\mu - 4 \end{aligned}$$

The complete solution, in vector form is:

$$\mathbf{x} = \begin{bmatrix} -4 \\ 0 \\ 4 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Our final example considers an echelon matrix which has a row consisting entirely of zeros.

Example 15 *Solve the system* $\mathbf{Ax} = \mathbf{b}$*, where*

$$\mathbf{A} = \begin{bmatrix} 2 & -4 & 1 & -8 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ 20 \\ 0 \end{bmatrix}$$

The last equation says that $0 = 0$*, so we only have to solve the upper block. This is exactly the system solved in the previous example, and the solution is stated there.*

Note that if the third component of $\mathbf{b}$ *had been a non-zero number, then there would be no solution.*

8 Gaussian elimination

We now turn to the general system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is an $m \times n$ matrix. There are two trivial cases:

- $\mathbf{A} = \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$: there is no solution
- $\mathbf{A} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$: every n-vector is a solution.

For the rest of the section, we ignore these trivial cases and assume that $\mathbf{A}$ has at least one non-zero entry.

To solve the system we apply *Gaussian elimination* on the rows of the *augmented matrix* $\begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$. We apply elementary row operations on the augmented matrix until we obtain a matrix $\begin{bmatrix} \mathbf{E} & \mathbf{c} \end{bmatrix}$ where $\mathbf{E}$ is an echelon matrix.

There are three types of *elementary row operations*:

1. Interchange of any two rows in the matrix.
2. Multiplication (or division) of a row by any scalar $k \neq 0$.
3. Addition of "k times any row" to another row.

Example 16 *Solve the system* $\mathbf{Ax} = \mathbf{b}$*, where*

$$\mathbf{A} = \begin{bmatrix} 2 & -4 & 1 & -8 \\ 4 & -8 & 7 & -6 \\ -1 & 2 & 1 & 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ 12 \\ 8 \end{bmatrix}$$

Start with the augmented matrix $\begin{bmatrix}\mathbf{A} & \mathbf{b}\end{bmatrix}$:

$$\left[\begin{array}{cccc|c} 2 & -4 & 1 & -8 & -4 \\ 4 & -8 & 7 & -6 & 12 \\ -1 & 2 & 1 & 7 & 8 \end{array}\right]$$

1. (a) *Subtract* 2 *times the first row from the second row:*

$$\left[\begin{array}{cccc|c} 2 & -4 & 1 & -8 & -4 \\ 0 & 0 & 5 & 10 & 20 \\ -1 & 2 & 1 & 7 & 8 \end{array}\right]$$

   (b) *Add* 1/2 *times the first row to the third:*

$$\left[\begin{array}{cccc|c} 2 & -4 & 1 & -8 & -4 \\ 0 & 0 & 5 & 10 & 20 \\ 0 & 0 & 3/2 & 3 & 6 \end{array}\right]$$

2. (a) *Subtract* 3/10 *times the second row from the third:*

$$\left[\begin{array}{cccc|c} 2 & -4 & 1 & -8 & -4 \\ 0 & 0 & 5 & 10 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We have reduced our system of equations to one in which the coefficient is a row echelon matrix. This row echelon matrix is the same one as in the previous example and the full solution is presented there.

Example 17 *Solve the following system of equations by Gaussian elimination:*

$$\begin{aligned} x - 3y &= -20 \\ 2x + y - z &= -1 \\ 4x + 6z &= 10 \end{aligned}$$

In matrix form:

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ 2 & 1 & -1 \\ 4 & 0 & 6 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -20 \\ -1 \\ 10 \end{bmatrix}$$

The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & -20 \\ 2 & 1 & -1 & -1 \\ 4 & 0 & 6 & 10 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 0 & -20 \\ 0 & 7 & -1 & 39 \\ 0 & 12 & 6 & 90 \end{array}\right] \begin{array}{l} \\ (2) - 2 \times (1) \\ (3) - 4 \times (1) \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 0 & -20 \\ 0 & 1 & -1/7 & 39/7 \\ 0 & 12 & 6 & 90 \end{array}\right] (2) \times 1/7$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 0 & -20 \\ 0 & 1 & -1/7 & 39/7 \\ 0 & 0 & 54/7 & 162/7 \end{array}\right] (3) - 12 \times (2)$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 0 & -20 \\ 0 & 1 & -1/7 & 39/7 \\ 0 & 0 & 1 & 3 \end{array}\right] (3) \times 7/54$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 0 & -20 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right] (2) + 1/7 \times (3)$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right] (1) + 3 \times (2)$$

This is in reduced row echelon form. The solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}$

8.1 Number of equations, number of unknowns

Given a system of m linear equations in n unknowns, there are three possibilities:

1. there is a unique solution
2. there is no solution
3. there are infinitely many solutions

If

- $m \geq n$ then any one of these cases can occur
- $m < n$ then only the last two cases can occur.

9 Inverting a matrix

Definition 12 *A square matrix* $\mathbf{A}$ *is said to be invertible if there is a square matrix* $\mathbf{A}^{-1}$ *that satisfies the condition*

$$\mathbf{A}\mathbf{A}^{-\mathbf{1}} = \mathbf{A}^{-\mathbf{1}}\mathbf{A} = \mathbf{I}$$

The matrix $\mathbf{A}^{-1}$ *is called the* inverse matrix *of* $\mathbf{A}$.

9.1 General facts about inverses.

1. Not every square matrix has an inverse. If a square matrix $\mathbf{A}$ has an inverse, $\mathbf{A}$ is said to be *non-singular*; if $\mathbf{A}$ is not invertible, it is called a *singular* matrix.

   In order for an inverse to exist, a *necessary* condition is that the matrix be square.

   Once this necessary condition has been met, a *sufficient* condition for the non-singularity of a matrix is that its rows (or columns) be linearly independent.

   When the dual conditions of squareness and linear independence are taken together, they constitute the necessary-and-sufficient condition for nonsingularity (nonsingularity $\Leftrightarrow$ squareness and linear independence).

2. If $\mathbf{A}$ is invertible,so is $\mathbf{A}^{-1}$, and $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$, i.e. $\mathbf{A}$ and $\mathbf{A}^{-1}$ are inverses of each other.

3. If $\mathbf{A}$ and $\mathbf{B}$ are invertible, so is $\mathbf{AB}$, and

   $$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

   Note that since matrix multiplication is not commutative, $(\mathbf{AB})^{-1}$ is not in general equal to $\mathbf{A}^{-1}\mathbf{B}^{-1}$.

4. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices such that $\mathbf{AB} = \mathbf{I}$, then both $\mathbf{A}$ and $\mathbf{B}$ are invertible, $\mathbf{A}^{-1} = \mathbf{B}$ and $\mathbf{B}^{-1} = \mathbf{A}$.

9.1.1 Some important points

1. The adjectives singular, non-singular and invertible apply only to square matrices.

2. $\mathbf{AB}^{-1}$ always means $\mathbf{A}\left(\mathbf{B}^{-1}\right)$, not $(\mathbf{AB})^{-1}$.

3. Inverse matrices bear some resemblance to reciprocals, but it should not be taken too literally. In particular if $\mathbf{A} \neq \mathbf{B}$ then $\mathbf{AB}^{-1}$ and $\mathbf{B}^{-1}\mathbf{A}$ are in general different; it is therefore not a good idea to denote either of these matrices by $\dfrac{\mathbf{A}}{\mathbf{B}}$. Remember, there is no such thing as matrix division.

9.2 Application and calculation

Inverses provide us with a method for solving systems of equations.

Given the system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is an invertible $n \times n$ matrix and $\mathbf{b}$ is an n-vector, then

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

is the unique solution.

We could still use Gaussian elimination to solve the system of equations. However, knowing $\mathbf{A}^{-1}$ enables us to solve $\mathbf{Ax} = \mathbf{b}$ for *any* vector $\mathbf{b}$, and this can be useful.

To invert a matrix $\mathbf{A}$, set up an augmented matrix with $\mathbf{A}$ on the left and the identity matrix on the right: $\begin{bmatrix}\mathbf{A} & \mathbf{I}\end{bmatrix}$.

Then apply row operations until the coefficient matrix on the left is reduced to an identity matrix. The matrix on the right will be the inverse: $\begin{bmatrix}\mathbf{I} & \mathbf{A}^{-1}\end{bmatrix}$.

Example 18 *Invert the matrix*

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Start with the augmented matrix $\begin{bmatrix}\mathbf{A} & \mathbf{I}\end{bmatrix}$:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Now reduce the matrix on the left to an identity matrix by applying the elementary row operations:

1. *Subtract* $3/2$ *times row 1 from rows 2 and 3:*

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -2 & -\frac{3}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & -1 & -\frac{3}{2} & 0 & 1 \end{array}\right]$$

2. *Add twice the second row to the first row, and subtract the second row from the third row:*

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -2 & -2 & 2 & 0 \\ 0 & -\frac{1}{2} & -2 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

3. Add twice the third row to the first and second rows:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -2 & 0 & 2 \\ 0 & -\frac{1}{2} & 0 & -\frac{3}{2} & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

4. Divide the first row by 2*, and multiply the second row by* $-2:$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 3 & 2 & -4 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Therefore,

$$\mathbf{A}^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 2 & -4 \\ 0 & -1 & 1 \end{bmatrix}$$

Example 19 *Solve the following system of equations using the inverse method:*

$$\begin{aligned} 2x + y + 2z &= 1 \\ 3x + y + z &= 2 \\ 3x + y + 2z &= 3 \end{aligned}$$

Solve using $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
The coefficient matrix is :

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

We calculated the inverse in the last example:

$$\mathbf{A}^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 2 & -4 \\ 0 & -1 & 1 \end{bmatrix}$$

So

$$\begin{aligned} \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\ &= \begin{bmatrix} -1 & 0 & 1 \\ 3 & 2 & -4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \end{aligned}$$

Example 20 *Solve the following system of equations using the inverse method:*

$$\begin{aligned} 2x - y + z &= 17 \\ x - 2y - z &= 4 \\ 3x + y - 2z &= -9 \end{aligned}$$

Solve using $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
The coefficient matrix is

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -2 & -1 \\ 3 & 1 & -2 \end{bmatrix}$$

We need to find $\mathbf{A}^{-1}$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 3 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 \\ 3 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \quad (1) \leftrightarrow (2)$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & -2 & 0 \\ 0 & 7 & 1 & 0 & -3 & 1 \end{array}\right] \quad \begin{array}{c} \\ (2) - 2 \times (1) \\ (3) - 3 \times (1) \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1/3 & -2/3 & 0 \\ 0 & 7 & 1 & 0 & -3 & 1 \end{array}\right] \quad (2) \times 1/3$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2/3 & -1/3 & 0 \\ 0 & 1 & 1 & 1/3 & -2/3 & 0 \\ 0 & 0 & -6 & -7/3 & 5/3 & 1 \end{array}\right] \quad \begin{array}{c} (1) + 2 \times (2) \\ \\ (3) - 7 \times (2) \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2/3 & -1/3 & 0 \\ 0 & 1 & 1 & 1/3 & -2/3 & 0 \\ 0 & 0 & 1 & 7/18 & -5/18 & -1/6 \end{array}\right] \quad (3) \times (-1/6)$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/18 & -1/18 & 1/6 \\ 0 & 1 & 0 & -1/18 & -7/18 & 1/6 \\ 0 & 0 & 1 & 7/18 & -5/18 & -1/6 \end{array}\right] \quad \begin{array}{c} (1) - (3) \\ (2) - (3) \\ \\ \end{array}$$

Therefore

$$
\begin{aligned}
\mathbf{A}^{-1} &= \begin{bmatrix} 5/18 & -1/18 & 1/6 \\ -1/18 & -7/18 & 1/6 \\ 7/18 & -5/18 & -1/6 \end{bmatrix} \\
&= \frac{1}{18}\begin{bmatrix} 5 & -1 & 3 \\ -1 & -7 & 3 \\ 7 & -5 & -3 \end{bmatrix}
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\
&= \frac{1}{18}\begin{bmatrix} 5 & -1 & 3 \\ -1 & -7 & 3 \\ 7 & -5 & -3 \end{bmatrix}\begin{bmatrix} 17 \\ 4 \\ -9 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}
\end{aligned}
$$

10 Linear dependence and rank

An $n \times n$ coefficient matrix $\mathbf{A}$ can be considered as an ordered set of row vectors. For the rows (row vectors) to be linearly independent, none must be a linear combination of the rest. More formally:

$\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^n$ are linearly independent if the only scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, that satisfy the vector equation

$$\alpha_1\mathbf{a}^1 + \alpha_2\mathbf{a}^2 + \ldots + \alpha_n\mathbf{a}^n = \mathbf{0}$$

be $\alpha_i = 0$ for all i.

Example 21 *If the coefficient matrix is*

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 8 & 10 \end{bmatrix} = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \mathbf{a}^3 \end{bmatrix}$$

then, since $\begin{bmatrix} 6 & 8 & 10 \end{bmatrix} = 2\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$, *we have* $2\mathbf{a}^1 + 0\mathbf{a}^2 - \mathbf{a}^3 = \mathbf{0}$.

Thus the third row is expressible as a linear combination of the first two, and the rows are not linearly independent. (Equivalently, since the set of scalars that led to the zero vector is not $\alpha_i = 0$ *for all* i*, it follows that the rows are linearly dependent.)*

10.1 Rank of a matrix

The rank of an $m \times n$ matrix $\mathbf{A}$ is the maximum number of linearly independent rows that can be found in the matrix. (The rank also tells us the maximum number of linearly independent columns that can be found in the matrix $\mathbf{A}$). The rank of an $m \times n$ matrix can be at most m or n, whichever is smaller.

10.1.1 Finding the rank

One method for finding the rank of a matrix $\mathbf{A}$ (not necessarily square), i.e. for determining the number of independent rows in $\mathbf{A}$, involves transforming $\mathbf{A}$ into an echelon matrix by Gaussian reduction.

Then

rank = number of pivots = number of non-zero rows = number of basic columns

Example 22 *Find the rank of the matrix*

$$\mathbf{A} = \begin{bmatrix} 0 & -11 & -4 \\ 2 & 6 & 2 \\ 4 & 1 & 0 \end{bmatrix}$$

This can be transformed into the row echelon matrix (show this):

$$\begin{bmatrix} 1 & 0 & -\frac{1}{11} \\ 0 & 1 & \frac{4}{11} \\ 0 & 0 & 0 \end{bmatrix}$$

Since this matrix contains two pivots (two non-zero rows, two basic columns) we can conclude that $rank(\mathbf{A}) = 2$.

10.1.2 Note

For an $n \times n$ matrix $\mathbf{A}$,
$\mathbf{A}$ is non-singular
$\Leftrightarrow$ $\mathbf{A}$ has n linearly independent rows (or columns)
$\Leftrightarrow$ $\mathbf{A}$ is of rank n
$\Leftrightarrow$ its echelon matrix must contain exactly n non-zero rows, with no zero rows at all
$\Leftrightarrow$ a unique solution to the system $\mathbf{Ax} = \mathbf{b}$ exists, given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
$\Leftrightarrow$ its determinant is non-zero.

In our previous example $rank(\mathbf{A}) = 2 < 3 = n$. Hence, $\mathbf{A}$ is singular.

11 Determinants

Determinants are defined only for *square* matrices. It is denoted by $|\mathbf{A}|$ (or $\det \mathbf{A}$) and is a uniquely defined scalar (number) associated with that matrix. (Do not confuse this notation with the symbol for the absolute value of a number!)

11.1 Calculating determinants

11.1.1 Using the rules

The following rules apply to determinants:

1. The determinant of a triangular matrix is the product of its diagonal entries.
2. The determinant changes sign when two rows are exchanged.
3. If a multiple of one row is subtracted from another row, the determinant remains unchanged.

To calculate the determinant we apply Gaussian elimination, reducing the given matrix $\mathbf{A}$ to an upper triangular matrix $\mathbf{U}$. $\det \mathbf{U}$ is the product of the diagonal entries of $\mathbf{U}$. And

$$\det \mathbf{A} = (-1)^k \det \mathbf{U}$$

where k is the number of times two rows have been exchanged in the reduction process.

Example 23

$$\begin{vmatrix} 0 & 0 & 6 \\ 0 & 4 & 5 \\ 1 & 2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = -24$$

Example 24 *Calculate* $\det \mathbf{A}$ *when*

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 7 \\ 5 & 8 & 6 \end{bmatrix}$$

The elimination step yields the matrix:

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 3 \\ 0 & \frac{1}{2} & -4 \end{bmatrix}$$

Exchanging the second and third rows gives:

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & \frac{1}{2} & -4 \\ 0 & 0 & 3 \end{bmatrix} = \mathbf{U}$$

Since there has been one row exchange,

$$\det \mathbf{A} = (-1)^1 \det \mathbf{U} = -\left(2 \times \frac{1}{2} \times 3\right) = -3$$

11.1.2 Other methods

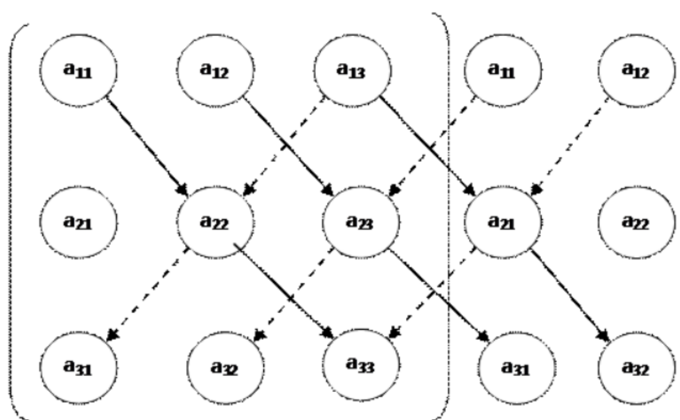
We can use the rules to derive a formula for the determinant of a 2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

We can also derive a formula for the determinant of a 3×3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Fortunately, there is a simple way to apply this formula:



Example 25

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (2)(5)(9) + (1)(6)(7) + (3)(8)(4) - (2)(8)(6) - (1)(4)(9) - (3)(5)(7) = -9.$$

This method of cross-diagonal multiplication is NOT applicable to determinants of orders higher than 3. We must use the Laplace expansion of the determinant instead.

11.1.3 Laplace Expansion

We begin by explaining Laplace expansion for a third-order determinant.

Consider the 3×3 matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

For each $i = 1, 2, 3$ and $j = 1, 2, 3$ we define the (i, j) *cofactor* of $\mathbf{A}$ to be the scalar

$$c_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij}$$

where $\mathbf{A}_{ij}$ is the 2×2 matrix obtained from $\mathbf{A}$ by deleting its ith row and jth column. For instance:

$$\begin{aligned} c_{11} &= +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ c_{12} &= -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \end{aligned}$$

We can now express a third-order determinant as

$$\begin{aligned} |\mathbf{A}| &= a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} \\ &= \sum_{j=1}^{3} a_{1j}c_{1j} \end{aligned}$$

i.e. as a sum of three terms, each of which is the product of a first-row element and its corresponding cofactor.

It is also feasible to expand a determinant by any row or any column of the matrix. The easiest way to expand the determinant is to expand by a row or column with some zero entries.

Example 26 *Compute*

$$|\mathbf{A}| = \begin{vmatrix} 5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0 \end{vmatrix}$$

We can expand by the first row:

$$|\mathbf{A}| = 5\begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} - 6\begin{vmatrix} 2 & 0 \\ 7 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} = -27$$

Or the first column:

$$|\mathbf{A}| = 5\begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} - 2\begin{vmatrix} 6 & 1 \\ -3 & 0 \end{vmatrix} + 7\begin{vmatrix} 6 & 1 \\ 3 & 0 \end{vmatrix} = -27$$

The easiest way is to expand by the third column:

$$|\mathbf{A}| = 1\begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} = -27$$

Example 27 *Compute*

$$|\mathbf{A}| = \begin{vmatrix} 4 & -3 & 4 & 6 \\ -1 & -3 & 0 & 3 \\ -4 & -4 & 2 & -2 \\ 4 & 3 & -3 & -4 \end{vmatrix}$$

Expanding by the third column:

$$\begin{aligned}
|\mathbf{A}| &= 4\begin{vmatrix} -1 & -3 & 3 \\ -4 & -4 & -2 \\ 4 & 3 & -4 \end{vmatrix} - 0\begin{vmatrix} 4 & -3 & 6 \\ -4 & -4 & -2 \\ 4 & 3 & -4 \end{vmatrix} + 2\begin{vmatrix} 4 & -3 & 6 \\ -1 & -3 & 3 \\ 4 & 3 & -4 \end{vmatrix} - (-3)\begin{vmatrix} 4 & -3 & 6 \\ -1 & -3 & 3 \\ -4 & -4 & -2 \end{vmatrix} \\
&= 4\left[-1\begin{vmatrix} -4 & -2 \\ 3 & -4 \end{vmatrix} - (-3)\begin{vmatrix} -4 & -2 \\ 4 & -4 \end{vmatrix} + 3\begin{vmatrix} -4 & -4 \\ 4 & 3 \end{vmatrix}\right] \\
&\quad +2\left[4\begin{vmatrix} -3 & 3 \\ 3 & -4 \end{vmatrix} - (-3)\begin{vmatrix} -1 & 3 \\ 4 & -4 \end{vmatrix} + 6\begin{vmatrix} -1 & -3 \\ 4 & 3 \end{vmatrix}\right] \\
&\quad +3\left[4\begin{vmatrix} -3 & 3 \\ -4 & -2 \end{vmatrix} - (-3)\begin{vmatrix} -1 & 3 \\ -4 & -2 \end{vmatrix} + 6\begin{vmatrix} -1 & -3 \\ -4 & -4 \end{vmatrix}\right] \\
&= 4\left[-1\,(22) + 3\,(24) + 3\,(4)\right] \\
&\quad +2\left[4\,(3) + 3\,(-8) + 6\,(9)\right] \\
&\quad +3\left[4\,(18) + 3\,(14) + 6\,(-8)\right] \\
&= 4\,(62) + 2\,(42) + 3\,(66) \\
&= 530
\end{aligned}$$

To sum up, the value of a determinant $|\mathbf{A}|$ of order n can be found by the Laplace expansion of any row or any column as follows:

$$\begin{aligned}
|\mathbf{A}| &= \sum_{j=1}^{n} a_{ij}c_{ij} && \text{[expansion by the } i\text{th row]} \\
&= \sum_{i=1}^{n} a_{ij}c_{ij} && \text{[expansion by the } j\text{th column]}
\end{aligned}$$

11.2 Properties of determinants

1. A square matrix is singular if and only if its determinant is zero. This provides us with another test for the singularity of a matrix. When a matrix has a non-zero determinant, we say that matrix is non-singular. This means that there is no linear dependence among the rows and columns of the matrix, and so there must be a unique solution to the system of equations that the coefficient matrix is representative of.

2. If *one* row of a matrix is multiplied by the scalar λ, and the *others are left unchanged*, the determinant is multiplied by λ.

3. If $\mathbf{A}$ is $n \times n$, $\det(\lambda \mathbf{A}) = \lambda^n \det \mathbf{A}$.

4. $\det(\mathbf{AB}) = \det \mathbf{A} \times \det \mathbf{B}$

5. $\det \mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}}$

12 Transposition

Definition 13 *The* transpose *of a (not necessarily square matrix)* $\mathbf{A}$ *is the matrix* $\mathbf{A}'$ *(or* $\mathbf{A}^T$*) whose rows are columns of* $\mathbf{A}$,

Example 28

$$\text{If } \mathbf{A} = \begin{bmatrix} 3 & 1 & 4 \\ 8 & 9 & 6 \end{bmatrix} \quad \text{then } \mathbf{A}' = \begin{bmatrix} 3 & 8 \\ 1 & 9 \\ 4 & 6 \end{bmatrix}$$

12.1 Properties of transposes

1. $(\alpha \mathbf{A} + \beta \mathbf{B})' = \alpha \mathbf{A}' + \beta \mathbf{B}'$
2. $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

   Note the order of multiplication. In particular note that $(\mathbf{AB})' \neq \mathbf{A}'\mathbf{B}'$ in general.
3. $(\mathbf{A}')' = \mathbf{A}$
4. $(\mathbf{A}')^{-1} = \left(\mathbf{A}^{-1}\right)'$
5. $\operatorname{rank}(\mathbf{A}') = \operatorname{rank}(\mathbf{A})$
6. $\det \mathbf{A}' = \det \mathbf{A}$

13 The adjoint matrix

Definition 14 *The* cofactor matrix *of an $n \times n$ matrix $\mathbf{A}$ is the matrix of cofactors: it is the matrix $\mathbf{C}$ whose $(i.j)$ entry c_{ij} is the $(i.j)$ cofactor of $\mathbf{A}$.*

Definition 15 *The* adjoint matrix *of $\mathbf{A}$ is the transpose of the cofactor matrix, and is denoted $adj\,\mathbf{A}$.*

$$adj\,\mathbf{A} = \mathbf{C}'$$

13.1 Finding the inverse of a matrix

We can now use a formula for the inverse of a matrix, often called the 'adj-over-det' formula: if $\mathbf{A}$ is invertible,

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} adj\,\mathbf{A}$$

Example 29 *Find the inverse of $\mathbf{A}$*

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}$$

1. *Find $|\mathbf{A}|$. (expand by first column)*

$$|\mathbf{A}| = 4 \begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} - 0 + 3 \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 99 \neq 0 \text{ so inverse exists.}$$

2. Find the cofactor matrix

$$\begin{aligned}
\mathbf{C} &= \begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 21 & 6 & -9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix}
\end{aligned}$$

3. Find the adjoint matrix

$$\begin{aligned}
adj\mathbf{A} &= \mathbf{C}' \\
&= \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}
\end{aligned}$$

4. Find the inverse

$$\begin{aligned}
\mathbf{A}^{-1} &= \frac{1}{|\mathbf{A}|} adj\mathbf{A} \\
&= \frac{1}{99} \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}
\end{aligned}$$

Example 30 *Use the inverse method to solve the system* $\mathbf{Ax} = \mathbf{b}$*, where*

$$\mathbf{A} = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}$$

We solve using $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.
Step 1*: Find the inverse of* $\mathbf{A}$.

$$\begin{aligned}
|\mathbf{A}| &= 6 \begin{vmatrix} 4 & -2 \\ -1 & 5 \end{vmatrix} - 3 \begin{vmatrix} 1 & -2 \\ 4 & 5 \end{vmatrix} + 1 \begin{vmatrix} 1 & 4 \\ 4 & -1 \end{vmatrix} \\
&= 6\,(20-2) - 3\,(5+8) + 1\,(-1-16) \\
&= 52 \\
&\neq 0 \therefore \text{ a unique solution exists.}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{C} &= \begin{bmatrix} \begin{vmatrix} 4 & -2 \\ -1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 4 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ -1 & 5 \end{vmatrix} & \begin{vmatrix} 6 & 1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 6 & 3 \\ 4 & -1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} & -\begin{vmatrix} 6 & 1 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 6 & 3 \\ 1 & 4 \end{vmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 18 & -13 & -17 \\ -16 & 26 & 18 \\ -10 & 13 & 21 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
adj\,\mathbf{A} &= \mathbf{C}' \\
&= \begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}^{-1} &= \frac{1}{|\mathbf{A}|} adj\,\mathbf{A} \\
&= \frac{1}{52} \begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}
\end{aligned}
$$

Step 2*: Now solve for* **x**.

$$
\begin{aligned}
\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\
&= \frac{1}{52} \begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix} \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}
\end{aligned}
$$

Example 31 *Consider the general* 2×2 *matrix*

$$
\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
$$

We can find the inverse of **A** *by using the 'adj-over-det' formula:*

$$
|\mathbf{A}| = ad - bc
$$

$$\begin{aligned} adj\mathbf{A} &= \mathbf{C}' \\ &= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}' \\ &= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{|\mathbf{A}|} adj\mathbf{A} \\ &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

This gives us a formula for inverting a 2×2 *matrix.*

13.2 Cramer's Rule

Note that we can write the inverse method in general form as:

$$\mathbf{x} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} \sum_{i=1}^{n} b_i c_{i1} \\ \sum_{i=1}^{n} b_i c_{i2} \\ \vdots \\ \sum_{i=1}^{n} b_i c_{in} \end{bmatrix}$$

This should look familiar to you. Recall that the expansion of a determinant $|\mathbf{A}|$ by its first column can be expressed in the form $\sum\limits_{i=1}^{n} a_{i1} c_{i1}$.

If we replace the first column of $|\mathbf{A}|$ by the column vector $\mathbf{b}$ but keep all the other columns intact, then a new determinant will result, which we can call $|\mathbf{A}_1|$ - the subscript 1 indicating that the first column has been replaced by $\mathbf{b}$. The expansion of $|\mathbf{A}_1|$ by its first column (the $\mathbf{b}$ column) will yield the expression $\sum\limits_{i=1}^{n} b_i c_{i1}$, because the elements b_i now take the place of the elements a_{i1}. We can therefore write

$$x_1^* = \frac{1}{|\mathbf{A}|} |\mathbf{A}_1|$$

This procedure can be generalised to give us Cramer's Rule:
If $\mathbf{A}$ is invertible, the solution to the system $\mathbf{Ax} = \mathbf{b}$ is

$$x_j^* = \frac{|\mathbf{A}_j|}{|\mathbf{A}|} \quad \text{for } j = 1, \ldots, n$$

where, for each $j, \mathbf{A}_j$ is the matrix obtained from $\mathbf{A}$ by replacing its jth column by $\mathbf{b}$.

Example 32 *Find the solution of the equation system*

$$\begin{aligned} 5x_1 + 3x_2 &= 30 \\ 6x_1 - 2x_2 &= 8 \end{aligned}$$

In matrix form:

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 30 \\ 8 \end{bmatrix} \end{aligned}$$

Step 1*: Find* $|\mathbf{A}|$.

$$|\mathbf{A}| = \begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix} = -28 \neq 0 \therefore \text{ a unique solution exists.}$$

Step 2*: Find* $|\mathbf{A}_1|$ *by replacing the first column of* $|\mathbf{A}|$ *with* $\mathbf{b}$.

$$|\mathbf{A}_1| = \begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix} = -84$$

Step 3*: Find* $|\mathbf{A}_2|$ *by replacing the second column of* $|\mathbf{A}|$ *with* $\mathbf{b}$.

$$|\mathbf{A}_2| = \begin{vmatrix} 5 & 30 \\ 6 & 8 \end{vmatrix} = -140$$

Step 4*: Find* x_1^* *and* x_2^*.

$$\begin{aligned} x_1^* &= \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{-84}{-28} = 3 \\ x_2^* &= \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{-140}{-28} = 5 \end{aligned}$$

Example 33 *Find the solution of the equation system*

$$\begin{aligned} 7x_1 - x_2 - x_3 &= 0 \\ 10x_1 - 2x_2 + x_3 &= 8 \\ 6x_1 + 3x_2 - 2x_3 &= 7 \end{aligned}$$

In matrix form:

$$\mathbf{A} = \begin{bmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 8 \\ 7 \end{bmatrix}$$

Step 1*: Find* $|\mathbf{A}|$.

$$\begin{aligned} |\mathbf{A}| &= 7\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} + 1\begin{vmatrix} 10 & 1 \\ 6 & -2 \end{vmatrix} - 1\begin{vmatrix} 10 & -2 \\ 6 & 3 \end{vmatrix} \\ &= -61 \neq 0 \therefore \text{ a unique solution exists.} \end{aligned}$$

Step 2*: Find* $|\mathbf{A}_1|$ *by replacing the first column of* $|\mathbf{A}|$ *with* $\mathbf{b}$.

$$|\mathbf{A}_1| = \begin{vmatrix} 0 & -1 & -1 \\ 8 & -2 & 1 \\ 7 & 3 & -2 \end{vmatrix} = 0 + 1\begin{vmatrix} 8 & 1 \\ 7 & -2 \end{vmatrix} - 1\begin{vmatrix} 8 & -2 \\ 7 & 3 \end{vmatrix} = -61$$

Step 3*: Find* $|\mathbf{A}_2|$ *by replacing the second column of* $|\mathbf{A}|$ *with* $\mathbf{b}$.

$$|\mathbf{A}_2| = \begin{vmatrix} 7 & 0 & -1 \\ 10 & 8 & 1 \\ 6 & 7 & -2 \end{vmatrix} = 7\begin{vmatrix} 8 & 1 \\ 7 & -2 \end{vmatrix} + 0 - 1\begin{vmatrix} 10 & 8 \\ 6 & 7 \end{vmatrix} = -183$$

Step 4*: Find* $|\mathbf{A}_3|$ *by replacing the second column of* $|\mathbf{A}|$ *with* $\mathbf{b}$.

$$|\mathbf{A}_3| = \begin{vmatrix} 7 & -1 & 0 \\ 10 & -2 & 8 \\ 6 & 3 & 7 \end{vmatrix} = 7\begin{vmatrix} -2 & 8 \\ 3 & 7 \end{vmatrix} + 1\begin{vmatrix} 10 & 8 \\ 6 & 7 \end{vmatrix} + 0 = -244$$

Step 5*: Solve for* x_1^*, x_2^* *and* x_3^*.

$$\begin{aligned} x_1^* &= \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{-61}{-61} = 1 \\ x_2^* &= \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{-183}{-61} = 3 \\ x_3^* &= \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{-244}{-61} = 4 \end{aligned}$$

14 Economic applications

We will use Cramer's Rule to solve some economic models. You could solve these using the inverse method as well, and you should make sure that you can do this.

Example 34 ***National Income Model***

Consider the simple national income model below

$$\begin{aligned} Y &= C + I + G \\ C &= a + bY \qquad\qquad a > 0, 0 < b < 1 \end{aligned}$$

where: Y represents national income, C represents (planned) consumption expenditure, I represents investment expenditure and G represents government expenditure.

These can be re-arranged as follows, with the endogenous variables on the left and the exogenous variables on the right:

$$\begin{aligned} Y - C &= I + G \\ -bY + C &= a \end{aligned}$$

In matrix form:

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I + G \\ a \end{bmatrix}$$

Use Cramer's Rule to solve for Y and C.

Step 1*: Find $|\mathbf{A}|$.*

$$\begin{aligned} |\mathbf{A}| = \begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix} &= 1 - b \\ &\neq 0 \quad \textit{(since } 0 < b < 1\textit{)} \\ &\therefore \quad \textit{a unique solution exists.} \end{aligned}$$

Step 2*: Find $|\mathbf{A}_1|$ by replacing the first column of $|\mathbf{A}|$ with $\mathbf{b}$.*

$$|\mathbf{A}_1| = \begin{vmatrix} I + G & -1 \\ a & 1 \end{vmatrix} = I + G + a$$

Step 3*: Solve for Y^*.*

$$Y^* = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{I + G + a}{1 - b}$$

Step 4*: Find $|\mathbf{A}_2|$ by replacing the second column of $|\mathbf{A}|$ with $\mathbf{b}$.*

$$|\mathbf{A}_2| = \begin{vmatrix} 1 & I + G \\ -b & a \end{vmatrix} = a + b\,(I + G)$$

Step 5*: Solve for* C^*.

$$C^* = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{a + b\,(I+G)}{1-b}$$

Example 35 ***National Income Model***

Let the national income model be:

$$\begin{aligned} Y &= C + I + G \\ C &= a + b(Y - T) \qquad\qquad a > 0, 0 < b < 1 \\ T &= d + tY \qquad\qquad\qquad\quad d > 0, 0 < t < 1 \end{aligned}$$

where Y *is national income,* C *is (planned) consumption expenditure,* I *is investment expenditure,* G *is government expenditure and* T *is taxes.*

Put system in matrix form:

$$\begin{bmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} I+G \\ a \\ d \end{bmatrix}$$

Step 1*: Find* $|\mathbf{A}|$.

$$\begin{aligned} |\mathbf{A}| = \begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix} &= 1\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0 \\ &= 1 - b + bt \\ &= 1 - b\,(1-t) \\ &> 0 \quad \textit{(since } 0 < b < 1, 0 < t < 1\textit{)} \\ &\therefore \quad \textit{a unique solution exists.} \end{aligned}$$

Step 2*: Find* $|\mathbf{A}_1|$ *by replacing the first column of* $|\mathbf{A}|$ *with* $\mathbf{b}$.

$$\begin{aligned} |\mathbf{A}_1| &= \begin{vmatrix} I+G & -1 & 0 \\ a & 1 & b \\ d & 0 & 1 \end{vmatrix} \\ &= 0 - b\begin{vmatrix} I+G & -1 \\ d & 0 \end{vmatrix} + 1\begin{vmatrix} I+G & -1 \\ a & 1 \end{vmatrix} \\ &= a - bd + I + G \end{aligned}$$

Step 3*: Solve for* Y^*.

$$Y^* = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{a - bd + I + G}{1 - b\,(1-t)}$$

Step 4*: Find* $|\mathbf{A}_2|$ *by replacing the second column of* $|\mathbf{A}|$ *with* $\mathbf{b}$.

$$\begin{aligned} |\mathbf{A}_2| &= \begin{vmatrix} 1 & I+G & 0 \\ -b & a & b \\ -t & d & 1 \end{vmatrix} \\ &= 0 - b \begin{vmatrix} 1 & I+G \\ -t & d \end{vmatrix} + 1 \begin{vmatrix} 1 & I+G \\ -b & a \end{vmatrix} \\ &= a - bd + b(1-t)(I+G) \end{aligned}$$

Step 5*: Solve for* C^*.

$$C^* = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{a - bd + b(1-t)(I+G)}{1 - b(1-t)}$$

Step 6*: Find* $|\mathbf{A}_3|$ *by replacing the third column of* $|\mathbf{A}|$ *with* $\mathbf{b}$.

$$\begin{aligned} |\mathbf{A}_3| &= \begin{vmatrix} 1 & -1 & I+G \\ -b & 1 & a \\ -t & 0 & d \end{vmatrix} \\ &= 1 \begin{vmatrix} -b & a \\ -t & d \end{vmatrix} + 1 \begin{vmatrix} 1 & I+G \\ -t & d \end{vmatrix} - 0 \\ &= d(1-b) + t(a+I+G) \end{aligned}$$

Step 7*: Solve for* T^*.

$$T^* = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{d(1-b) + t(a+I+G)}{1 - b(1-t)}$$

Example 36 ***Market Model***

The system of equations below describes the market for widgets:

$$\begin{aligned} Q_d &= \alpha - \beta P + \gamma G \\ Q_s &= -\delta + \theta P - \lambda N \\ Q_d &= Q_s \qquad\qquad \alpha, \beta, \gamma, \delta, \theta, \lambda > 0 \end{aligned}$$

where G *is the price of substitutes for widgets and* N *is the price of inputs used in producing widgets.*

Re-arrange equations:

$$\begin{aligned} Q_d + \beta P &= \alpha + \gamma G \\ Q_s - \theta P &= -\delta - \lambda N \\ Q_d - Q_s &= 0 \end{aligned}$$

Put into matrix form:

$$\begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & -\theta \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} = \begin{bmatrix} \alpha + \gamma G \\ -\delta - \lambda N \\ 0 \end{bmatrix}$$

Step 1*: Find* $|\mathbf{A}|$.

$$\begin{aligned} |\mathbf{A}| &= \begin{vmatrix} 1 & 0 & \beta \\ 0 & 1 & -\theta \\ 1 & -1 & 0 \end{vmatrix} \\ &= 1 \begin{vmatrix} 1 & -\theta \\ -1 & 0 \end{vmatrix} - 0 + 1 \begin{vmatrix} 0 & \beta \\ 1 & -\theta \end{vmatrix} \\ &= 1(0-\theta) + 1(0-\beta) \\ &= -(\beta+\theta) \end{aligned}$$

Step 2*: Find* $|\mathbf{A}_1|$.

$$\begin{aligned} |\mathbf{A}_1| &= \begin{vmatrix} \alpha + \gamma G & 0 & \beta \\ -\delta - \lambda N & 1 & -\theta \\ 0 & -1 & 0 \end{vmatrix} \\ &= 0 - \left(-1 \begin{vmatrix} \alpha + \gamma G & \beta \\ -\delta - \lambda N & -\theta \end{vmatrix} \right) + 0 \\ &= -\theta(\alpha + \gamma G) - \beta(-\delta - \lambda N) \\ &= -\theta(\alpha + \gamma G) + \beta(\delta + \lambda N) \end{aligned}$$

Step 3*: Solve for* Q_d.

$$\begin{aligned} Q_d &= \frac{|\mathbf{A}_1|}{|\mathbf{A}|} \\ &= \frac{-\theta(\alpha + \gamma G) + \beta(\delta + \lambda N)}{-(\beta + \theta)} \\ &= \frac{\theta(\alpha + \gamma G) - \beta(\delta + \lambda N)}{(\beta + \theta)} \\ &= Q_s \end{aligned}$$

Step 4*: Find* $|\mathbf{A}_2|$.

$$\begin{aligned}
|\mathbf{A}_2| &= \begin{vmatrix} 1 & \alpha+\gamma G & \beta \\ 0 & -\delta-\lambda N & -\theta \\ 1 & 0 & 0 \end{vmatrix} \\
&= 1\begin{vmatrix} \alpha+\gamma G & \beta \\ -\delta-\lambda N & -\theta \end{vmatrix} - 0 + 0 \\
&= -\theta\left(\alpha+\gamma G\right) - \beta\left(-\delta-\lambda N\right) \\
&= -\theta\left(\alpha+\gamma G\right) + \beta\left(\delta+\lambda N\right)
\end{aligned}$$

Step 5*: Solve for* Q_s.

$$\begin{aligned}
Q_s &= \frac{|\mathbf{A}_2|}{|\mathbf{A}|} \\
&= \frac{-\theta\left(\alpha+\gamma G\right) + \beta\left(\delta+\lambda N\right)}{-\left(\beta+\theta\right)} \\
&= \frac{\theta\left(\alpha+\gamma G\right) - \beta\left(\delta+\lambda N\right)}{\left(\beta+\theta\right)}
\end{aligned}$$

This is the same solution we had for Q_d*, which shouldn't surprise you because we know that* $Q_d = Q_s$*.*

Step 6*: Find* $|\mathbf{A}_3|$.

$$\begin{aligned}
|\mathbf{A}_3| &= \begin{vmatrix} 1 & 0 & \alpha+\gamma G \\ 0 & 1 & -\delta-\lambda N \\ 1 & -1 & 0 \end{vmatrix} \\
&= 1\begin{vmatrix} 1 & -\delta-\lambda N \\ -1 & 0 \end{vmatrix} - 0 + 1\begin{vmatrix} 0 & \alpha+\gamma G \\ 1 & -\delta-\lambda N \end{vmatrix} \\
&= 1\left(0-\delta-\lambda N\right) + 1\left(0-\alpha-\gamma G\right) \\
&= -\delta-\lambda N-\alpha-\gamma G
\end{aligned}$$

Step 7*: Solve for* P.

$$\begin{aligned}
P &= \frac{|\mathbf{A}_3|}{|\mathbf{A}|} \\
&= \frac{-\delta-\lambda N-\alpha-\gamma G}{-\left(\beta+\theta\right)} \\
&= \frac{\delta+\lambda N+\alpha+\gamma G}{\left(\beta+\theta\right)}
\end{aligned}$$

15 Quadratic forms

15.1 Quadratic forms and symmetric matrices

Recall that a quadratic function is a function of the form

$$f(x) = ax^2 + bx + c$$

where a, b, c are constants.

We can generalise this to a quadratic form in two variables x_1, x_2 :

$$g(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$$

By the rules of matrix multiplication, we may write

$$g(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & p \\ q & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where p and q are any two numbers such that $p + q = b$. Consider in particular what happens when $p = q = \frac{b}{2}$, then

$$g(x_1, x_2) = \mathbf{x}'\mathbf{A}\mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \mathbf{A} = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}$$

Note that $\mathbf{A}$ is a *symmetric matrix.*

Definition 16 *A* symmetric matrix *is a square matrix whose transpose is itself. So the* $n \times n$ *matrix* $\mathbf{A}$ *is symmetric if and only if*

$$\mathbf{A} = \mathbf{A}'$$

Definition 17 *Associated with each symmetric* $n \times n$ *matrix* $\mathbf{A}$ *is the* quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ *, where* $\mathbf{x}$ *is an arbitrary* n*-vector.*

Example 37 *Consider the typical* 2×2 *symmetric matrix*

$$\mathbf{A} = \begin{bmatrix} a & p \\ p & c \end{bmatrix}$$

which has the quadratic form

$$\mathbf{x}'\mathbf{A}\mathbf{x} = ax_1^2 + cx_2^2 + 2px_1x_2$$

Example 38 *Consider the typical* 3×3 *symmetric matrix*

$$\mathbf{A} = \begin{bmatrix} a & p & q \\ p & b & r \\ q & r & c \end{bmatrix}$$

which has the quadratic form

$$\mathbf{x}'\mathbf{A}\mathbf{x} = ax_1^2 + bx_2^2 + cx_3^2 + 2px_1x_2 + 2qx_1x_3 + 2rx_2x_3$$

15.2 Definite and semidefinite quadratic forms

Definition 18 *A symmetric matrix* $\mathbf{A}$ *is said to be*

$$\left.\begin{array}{c} \textit{Positive definite} \\ \textit{Positive semidefinite} \\ \textit{Negative definite} \\ \textit{Negative semidefinite} \end{array}\right\} \textit{if } \mathbf{x}'\mathbf{A}\mathbf{x} \textit{ is invariably} \left\{\begin{array}{cc} \textit{positive} & (>0) \\ \textit{nonnegative} & (\geq 0) \\ \textit{negative} & (<0) \\ \textit{nonpositive} & (\leq 0) \end{array}\right.$$

for every non-zero vector $\mathbf{x}$.

If $\mathbf{x}'\mathbf{A}\mathbf{x}$ changes sign when $\mathbf{x}$ assumes different values, $\mathbf{A}$ is said to be *indefinite*.

Example 39 *Let* $\mathbf{A}$ *be the symmetric matrix*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Then

$$\mathbf{x}'\mathbf{A}\mathbf{x} = x_1^2 + (x_2 - x_3)^2$$

which is nonnegative. Thus $\mathbf{A}$ *is positive semidefinite.*

Note that $\mathbf{A}$ *is not positive definite since* $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{0}$ *for any vector* $\mathbf{x}$ *of the form* $\begin{bmatrix} 0 \\ c \\ c \end{bmatrix}$, *such as* $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Example 40 *Let* $\mathbf{A}$ *be the symmetric matrix*

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$\mathbf{x}'\mathbf{A}\mathbf{x} = -x_1^2 + x_2^2$$

and

$$\begin{aligned} \mathbf{x}'\mathbf{A}\mathbf{x} &= -1 \text{ when } \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \mathbf{x}'\mathbf{A}\mathbf{x} &= 1 \text{ when } \mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore $\mathbf{A}$ *is indefinite.*

The terms 'positive definite', 'negative semidefinite', etc. are often applied to the quadratic forms themselves as well as the symmetric matrices which define them.

15.3 Testing symmetric matrices

The determinantal test for sign definiteness is widely used, although it is more easily applicable to positive and negative definiteness as opposed to semidefiniteness.

Before we can discuss the determinantal test for the sign definiteness of a matrix, we must first define a principal minor and a leading principal minor of a matrix.

Definition 19 *A* principal minor *of a square matrix* $\mathbf{A}$ *is the determinant of a submatrix obtained using the rule that the* k*th row of* $\mathbf{A}$ *is deleted if and only if the* k*th column of* $\mathbf{A}$ *is deleted.*

Definition 20 *A* leading principal minor *of a square matrix* $\mathbf{A}$ *is the determinant of a submatrix obtained by deleting everything except the first* m *rows and columns, for* $1 \leq m \leq n$.

Example 41 *Consider the* 3×3 *matrix* $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The first leading principal minor is the determinant of the submatrix obtained by deleting everything except the first 1 *row and column.*

$$\begin{aligned} |\mathbf{M}_1| &= \left| a_{11} \right| \\ &= a_{11} \end{aligned}$$

The second leading principal minor is the determinant of the submatrix obtained by deleting everything except the first 2 *rows and columns.*

$$\begin{aligned} |\mathbf{M}_2| &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

The third leading principal minor is the determinant of the submatrix obtained by deleting everything except the first 3 *rows and columns.*

$$\begin{aligned} |\mathbf{M}_3| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= |\mathbf{A}| \end{aligned}$$

The other principal minors are:

$$\begin{aligned} \begin{vmatrix} a_{22} \end{vmatrix} &= a_{22} \\ \begin{vmatrix} a_{33} \end{vmatrix} &= a_{33} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} &= a_{22}a_{33} - a_{23}a_{32} \\ \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} &= a_{11}a_{33} - a_{13}a_{31} \end{aligned}$$

We can now define a test for the sign definiteness of symmetric matrices:

1. An $n \times n$ symmetric matrix is positive definite if and only if its leading principal minors are all positive.

2. An $n \times n$ symmetric matrix is positive semidefinite if and only if its principal minors are all non-negative.

   Note that non-negativity of the leading principal minors is NOT a sufficient condition for a symmetric matrix to be positive semidefinite.

We can generalise these rules to tests for negative definiteness, but it is usually easier to consider the matrix $-\mathbf{A}$: $\mathbf{A}$ is negative definite if $-\mathbf{A}$ is positive definite, and $\mathbf{A}$ is negative semidefinite if $-\mathbf{A}$ is positive semidefinite.

1. An $n \times n$ symmetric matrix is negative definite if and only if its leading principal minors are alternately negative and positive, beginning with a negative (i.e. $|\mathbf{M}_1| < 0, |\mathbf{M}_2| > 0, |\mathbf{M}_3| < 0$, etc.)

Example 42 *Determine the definiteness of the symmetric matrix*

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 1 & 6 \end{bmatrix}$$

Let us first consider the leading principal minors:

$$\begin{aligned} |\mathbf{M}_1| &= |4| = 4 > 0 \\ |\mathbf{M}_2| &= \begin{vmatrix} 4 & 1 \\ 1 & 6 \end{vmatrix} = 23 > 0 \end{aligned}$$

The leading principal minors are all positive, therefore **A** *is positive definite.*

Example 43 *Determine the definiteness of the symmetric matrix*

$$\mathbf{A} = \begin{bmatrix} -2 & -1 & 2 \\ -1 & -2 & 2 \\ 2 & 2 & -5 \end{bmatrix}$$

Let us first consider the leading principal minors:

$$\begin{aligned} |\mathbf{M}_1| &= |-2| = -2 < 0 \\ |\mathbf{M}_2| &= \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = 3 > 0 \\ |\mathbf{M}_3| &= \begin{vmatrix} -2 & -1 & 2 \\ -1 & -2 & 2 \\ 2 & 2 & -5 \end{vmatrix} = -7 < 0 \end{aligned}$$

The leading principal minors alternate in sign beginning with a negative, therefore **A** *is negative definite.*

Alternatively, we could consider $-\mathbf{A}$:

$$-\mathbf{A} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 5 \end{bmatrix}$$

The leading principal minors are:

$$\begin{aligned} |\mathbf{M}_1| &= |2| = 2 > 0 \\ |\mathbf{M}_2| &= \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0 \\ |\mathbf{M}_3| &= \begin{vmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 5 \end{vmatrix} = 7 > 0 \end{aligned}$$

The leading principal minors are all positive, therefore $-\mathbf{A}$ *is positive definite. This means that* **A** *is negative definite.*

Example 44 *Determine the definiteness of the symmetric matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Let us first consider the leading principal minors:

$$\begin{aligned} |\mathbf{M}_1| &= |1| = 1 > 0 \\ |\mathbf{M}_2| &= \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \\ |\mathbf{M}_3| &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 0 \end{aligned}$$

The leading principal minors are not all positive, so we must also consider the other principal minors:

$$\begin{aligned} |0| &= 0 \\ |-1| &= -1 < 0 \\ \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} &= -1 < 0 \\ \begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} &= 0 \end{aligned}$$

The principal minors are not all non-negative so **A** *is not positive semidefinite. Even though the leading principal minors are non-negative, this is not sufficient to conclude that* **A** *is positive semidefinite.*

−**A** *is neither positive definite nor positive semidefinite, so* **A** *is also not negative definite or negative semidefinite.*

A *is therefore indefinite.*

References

[1] Chiang, A.C. and Wainwright, K. 2005. *Fundamental Methods of Mathematical Economics*, 4th ed. McGraw-Hill International Edition.

[2] Pemberton, M. and Rau, N.R. 2001. *Mathematics for Economists: An introductory textbook*, Manchester: Manchester University Press.