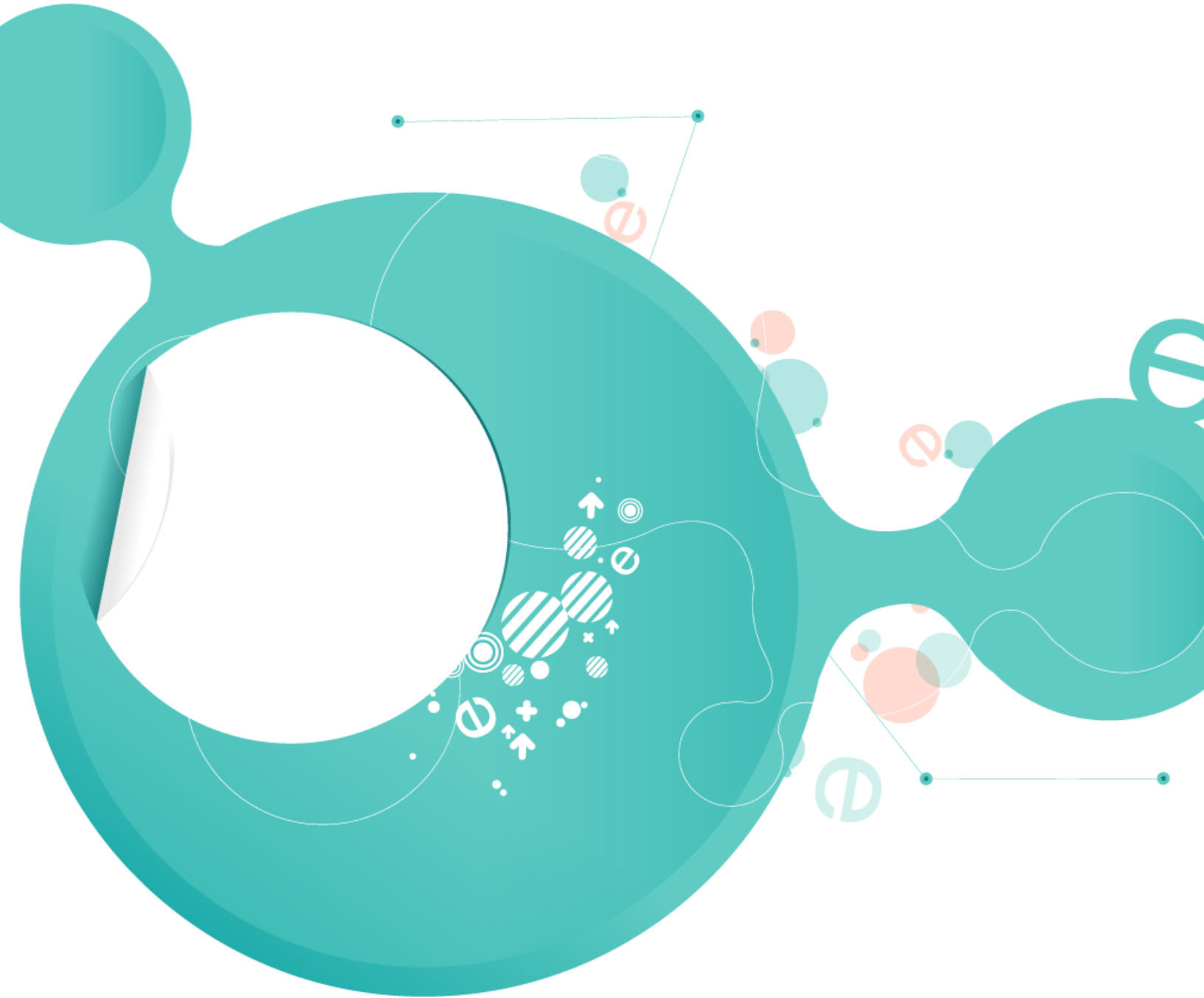




# Mathematics for Economists

## Revision of Basic Concepts



This work is licensed under a [Creative Commons Attribution-NonCommercial-ShareAlike 2.5 South Africa License](https://creativecommons.org/licenses/by-nc-sa/2.5/).

# Section 0: Revision of Basic Concepts

ECO4112F 2011

We begin by revising some basic mathematical concepts that should be familiar to you from previous mathematics courses.

## 1 Ingredients of a Mathematical Model

- *Variable*: Something whose magnitude can change e.g. price, profit, investment, consumption.
- *Endogenous variables*: The value of these variables originates from within the model. In other words, we need to solve the model to find out what the value of these variables are.
- *Exogenous variables*: The value of these variables is determined outside of the model. Often, exogenous variables are denoted by the use of the subscript 0. For example, in the national income model, where investment and government expenditure are exogenous (or given), we often denote them as  $I_0$  and  $G_0$ .
- *Constant*: A value that does not change.
- *Parameter*: A symbolic constant attached to a variable. For example, we can let  $a$  stand for a given constant and use the expression  $aP$  in a model (where  $P$  refers to the variable price). The symbol  $a$  represents a given constant but it can take virtually any value, and so is called a parametric constant (or simply a *parameter*).
- *Equation*: An expression that relates variables to each other. There are three main types of equations:

- *Definitional equation*: This sets up an identity between two different expressions that have the identical meaning. For example:

$$\pi \equiv TR - TC$$

We read this as: profit is identically equal to total revenue minus total cost.

An identity will always hold true in every situation, by definition.

- *Behavioural equation*: This specifies the way in which one variable behaves in response to changes in other variables. We use these kinds of equations to describe the general setting of the model we are working with. For example:

$$C = 80 + 3Q$$

Here, we show that as  $Q$  changes,  $C$  will change too.

- *Conditional equation*: This states a requirement to be satisfied. For example, in a model involving the idea of equilibrium, we must set up an equilibrium condition which tells us the conditions that need to be met if equilibrium is to occur. The one you are most familiar with comes from the model of a demand and supply, where

$$Q_d = Q_s$$

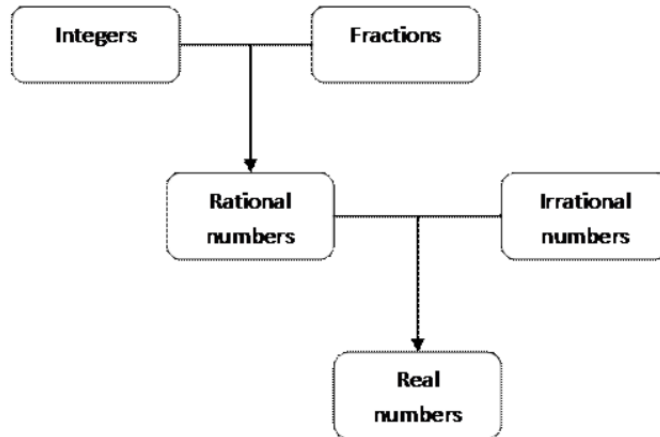
Similarly, an optimisation model either derives or applies one or more optimisation conditions. For example, in the theory of the firm we have the familiar condition

$$MR = MC$$

## 2 The Real Number System

- *Integers*: The set of all positive and negative whole numbers and the number zero (e.g. -1,-2,4,6), denoted by  $\mathbb{Z}$ .
- *Fractions*: Ratios of two integers, e.g.  $\frac{2}{3}, \frac{7}{5}, -\frac{5}{4}$ . Can be positive or negative.
- *Rational numbers*: Numbers that can be expressed as ratios of two integers. The set of all integers and the set of all fractions together form the set of all rational numbers, denoted by  $\mathbb{Q}$ .
- *Irrational numbers*: Numbers that cannot be expressed as ratios of two integers, e.g.  $\sqrt{2}, \pi$
- *Real numbers*: The set of all rational and irrational numbers, denoted by  $\mathbb{R}$ .

We can summarise the relationship between all the number sets using the following diagram:



### 3 The Concept of Sets

A *set* is simply a collection of distinct objects. The objects in a set are called the *elements* of the set.

We may write a set by *enumeration* or by *description*.

- Enumeration e.g. If we let  $S$  represent the set of the numbers 2, 3 and 4, we can write  $S = \{2, 3, 4\}$ .
- Description e.g. If we let  $J$  denote the set of all real numbers greater than 2 but less than 5, enumeration is difficult, and we describe the elements by  $J = \{x \mid 2 < x < 5\}$ . We read this as " $J$  is the set of all real numbers  $x$ , such that  $x$  is greater than 2 but less than 5".

A set with a finite number of elements is called a *finite set*, e.g. set  $S$ , while a set with an infinite number of elements is called an *infinite set*, e.g. set  $J$ .

Finite sets are always countable, i.e. their elements can be counted one by one in the sequence  $1, 2, 3, \dots$ . Infinite sets may, however, be either countable (e.g.  $\mathbb{Z}$ ) or uncountable (e.g.  $J$ ).

Membership in a set is indicated by the symbol  $\in$ , which is read as "is an element of". For example  $2 \in S, 3 \in J$ . The statement " $x$  is some real number" can be expressed by  $x \in \mathbb{R}$ .

## 4 Necessary versus Sufficient Conditions

The concepts of *necessary condition* and *sufficient condition* are frequently used in economics and it is important that we understand their precise meanings.

### 4.1 Necessary

A necessary condition is a prerequisite: Suppose that a statement  $p$  is true *only if* another statement  $q$  is true, then  $q$  constitutes a necessary condition of  $p$ . Symbolically we write this as

$$p \Rightarrow q$$

which is read as " $p$  only if  $q$ " or "if  $p$ , then  $q$ ".

It is also correct to interpret it as " $p$  implies  $q$ ".

We may also have  $p \Rightarrow w$  at the same time. In this case both  $q$  and  $w$  are necessary conditions for  $p$ .

**Example 1** Let  $p =$  "a person is a father" and  $q =$  "a person is male".

Then the statement  $p \Rightarrow q$  applies. A person is a father only if he is male, and to be male is a necessary condition for fatherhood. Note, however, that the converse is not true: fatherhood is not a necessary condition for maleness.

### 4.2 Sufficient

If a statement  $p$  is true if  $q$  is true, but  $p$  can also be true when  $q$  is not true, then  $q$  is said to be a sufficient condition for  $p$ . The truth of  $q$  suffices to establish the truth of  $p$ , but it is not a necessary condition for  $p$ . This is expressed symbolically as

$$p \Leftarrow q$$

which is read as " $p$  if  $q$ " (note the absence of the word only) or "if  $q$ , then  $p$ ".

It can also be interpreted to mean " $q$  implies  $p$ ".

**Example 2** Let  $p =$  "one can get to Europe" and  $q =$  "one takes a plane to Europe".

Then the statement  $p \Leftarrow q$  applies. Flying can get one to Europe, but since it's also possible to get there by ship, flying is not necessary. We can write  $p \Leftarrow q$  but not  $p \Rightarrow q$ .

### 4.3 Necessary and sufficient

In this situation  $q$  is both necessary and sufficient for  $p$ , and we write this as

$$p \Leftrightarrow q$$

which is read as " $p$  if and only if  $q$ " (also written as " $p$  iff  $q$ ").

This states that  $p$  implies  $q$  and that  $q$  implies  $p$ .

**Example 3** Let  $p =$  "there are less than 30 days in the month" and  $q =$  "it is the month of February".

Then the statement  $p \Leftrightarrow q$  applies. To have less than 30 days in the month, it is necessary that it be February ( $p \Rightarrow q$ ). Conversely, if the month of February is specified it is sufficient to establish that there are less than 30 days in the month ( $p \Leftarrow q$ ). Thus  $q$  is a necessary and sufficient condition for  $p$ .

**Example 4** In the following paired statements, let  $p$  be the first statement and  $q$  the second. Indicate for each case whether  $p \Rightarrow q$ ,  $p \Leftarrow q$ ,  $p \Leftrightarrow q$  or whether none of these apply.

1. Today is Wednesday. It is now later than Tuesday and earlier than Thursday.

For today to be Wednesday, it is necessary that it is now later than Tuesday and earlier than Thursday, so  $p \Rightarrow q$ .

If it is now later than Tuesday and earlier than Thursday, then it is sufficient to conclude that today is Wednesday, so  $p \Leftarrow q$ .

Thus  $q$  is a necessary and sufficient condition for  $p$ , i.e.  $p \Leftrightarrow q$ .

2. Alice has four apples and three bananas. Alice has some fruit.

For Alice to have four apples and three bananas, it is necessary that Alice has some fruit, so  $p \Rightarrow q$ .

If Alice has some fruit, then it is not sufficient to conclude that Alice has four apples and three bananas, so  $p \not\Leftarrow q$ .

Thus  $q$  is necessary, but not sufficient, for  $p$ , i.e.  $p \Rightarrow q$

3. Pianos are stringed instruments. Pianos and violins are stringed instruments.

For pianos to be stringed instruments, it is not necessary that pianos and violins be stringed instruments, so  $p \not\Rightarrow q$ .

If pianos and violins are stringed instruments, then it is sufficient to conclude that pianos are stringed instruments, so  $p \Leftarrow q$ .

Thus  $q$  is sufficient, but not necessary, for  $p$ , i.e.  $p \Leftarrow q$

## 5 Relations and Functions

An *ordered pair* associates a  $y$  value with an  $x$  value denoted by  $(x, y)$ . Note that  $(x, y) \neq (y, x)$  unless  $x = y$ .

*Relation*: Any collection of ordered pairs. Given an  $x$  value, one or more  $y$  values will be specified by that relation, e.g. the set  $\{(x, y) \mid y \leq x\}$  constitutes a relation.

When an  $x$  value is given it may not always be possible to determine a unique  $y$  value from a relation, e.g. if  $x = 1$ ,  $y$  can take on various values such as 0, 1, or  $-4$  and still satisfy the relation  $\{(x, y) \mid y \leq x\}$ .

*Function:* A set of ordered pairs with the property that any  $x$  value *uniquely* determines a  $y$  value, e.g.  $\{(x, y) \mid y = 2x\}$ . We say  $y$  is a function of  $x$  and denote this by  $y = f(x)$ , which is read as "y equals f of x".

For every  $x$ , there is one unique  $y$  value. However, the converse does not hold true. It need not be the case that for every  $y$  value, there is a unique  $x$ .

In the function  $y = f(x)$ ,  $y$  is the *dependent variable*, and  $x$  is the *independent variable*.

The set of all possible values that  $x$  can take is called the *domain*. The set of all possible values that  $y$  can take is called the *range*.

## 6 Types of Function

We consider specific types of function, each representing a different rule of mapping

### 6.1 Constant Functions

A function whose range consists of only one element is called a *constant function*, e.g.  $y = f(x) = 7$ . In the Cartesian plane, such a function will appear as a horizontal straight line.

### 6.2 Polynomial Functions

A polynomial function is simply a function with more than one term in it. It takes the general form:

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

in which each term contains a coefficient as well as a non-negative integer power of the variable  $x$ . The superscript indicators of the powers of  $x$  are called exponents. The highest power involved (i.e. the value of  $n$ ) is called the *degree* of the polynomial.

There are several subclasses of polynomial function, depending on the value of the exponents:

Case of $n = 0$ :	$y = a_0$	Constant function
Case of $n = 1$ :	$y = a_0 + a_1x$	Linear function (first degree polynomial)
Case of $n = 2$ :	$y = a_0 + a_1x + a_2x^2$	Quadratic function (second degree polynomial)
Case of $n = 3$ :	$y = a_0 + a_1x + a_2x^2 + a_3x^3$	Cubic function (third degree polynomial)

### 6.3 Rational Functions

A function in which  $y$  is expressed as a ratio of two polynomials in the variable  $x$  is known as a *rational function*, e.g.  $y = \frac{x-1}{x^2+2x+4}$ . Any polynomial function must be a rational function.

A special rational function is the rectangular hyperbola:  $y = \frac{a}{x}$  or  $xy = a$ .

### 6.4 Non Algebraic Functions

Any function expressed in terms of polynomials and/or roots (such as square root) of polynomials is an *algebraic function*. All the functions discussed so far are algebraic.

However, *exponential functions* such as  $y = b^x$ , in which the independent variable appears in the exponent, are non algebraic. *Logarithmic functions*, such as  $y = \log_b x$ , are also non algebraic. *Trigonometric functions* are also non algebraic. Nonalgebraic functions are also known as transcendental functions.

## 7 Exponents

For any natural number  $n$ ,

$$x^n = \underbrace{x \times x \times \dots \times x}_{n \text{ times}}$$

We refer to  $n$  as the *exponent* or *power* or *index*.

### 7.1 Rules

1.  $x^m \times x^n = x^{m+n}$
2.  $x^m / x^n = x^{m-n}$
3.  $x^{-n} = 1/x^n$
4.  $x^0 = 1$
5.  $(x^m)^n = x^{mn}$
6.  $(xy)^n = x^n y^n$

Note

- If  $x > 0$  and  $n$  is a positive integer,  $x^{1/n}$  is defined as the positive  $n$ th root of  $x$ , e.g.  $x^{1/2} = \sqrt{x}$ ,  $27^{1/3} = \sqrt[3]{27} = 3$ ,  $32^{1/5} = \sqrt[5]{32} = 2$ .



- Rule 3 provides a convenient notation for very small numbers, e.g.  $0.000000321 = 3.21 \times 10^{-7}$ .

**Example 5**

$$\begin{aligned} \frac{x^8 \times x^{-9}}{x^{-4}} &= \frac{x^{8-9}}{x^{-4}} && \text{(by rule 1)} \\ &= x^{-1-(-4)} && \text{(by rule 2)} \\ &= x^3 \end{aligned}$$

**Example 6**

$$\begin{aligned} \frac{x^{1/3} \times x^{1/6}}{x^{3/2}} &= \frac{x^{1/2}}{x^{3/2}} && \text{(by rule 1)} \\ &= x^{-1} && \text{(by rule 2)} \\ &= \frac{1}{x} && \text{(by rule 3)} \end{aligned}$$

**Example 7**

$$\begin{aligned} (x^2y^2)^{-1/4} \sqrt{y} &= \left((xy)^2\right)^{-1/4} \sqrt{y} && \text{(by rule 6)} \\ &= (xy)^{2 \times -1/4} \sqrt{y} && \text{(by rule 5)} \\ &= (xy)^{-1/2} \sqrt{y} \\ &= x^{-1/2} y^{-1/2} y^{1/2} && \text{(by rule 6)} \\ &= x^{-1/2} y^0 && \text{(by rule 1)} \\ &= x^{-1/2} \times 1 && \text{(by rule 4)} \\ &= \frac{1}{x^{1/2}} && \text{(by rule 3)} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

**Example 8**

$$\begin{aligned} (64x^6)^{-1/3} \times (9x)^{1/2} &= \left((2x)^6\right)^{-1/3} \times (9x)^{1/2} && \text{(by rule 6)} \\ &= (2x)^{-2} \times (9x)^{1/2} && \text{(by rule 5)} \\ &= \frac{3x^{1/2}}{4x^2} && \text{(by rules 6 and 3)} \\ &= \frac{3}{4x^{3/2}} \end{aligned}$$

## 7.2 Exponents and inequalities

If  $u, v, a$  are positive numbers such that  $u < v$ , then

- (a)  $u^a < v^a$  and  $u^{-a} > v^{-a}$
- (b)  $a^u < a^v$  if  $a > 1$  and  $a^u > a^v$  if  $a < 1$ .

## 8 Logarithms

For any given positive number  $y$ , there is exactly one real number  $x$  such that  $a^x = y$ . This  $x$  is called the *logarithm* of  $y$  to *base*  $a$ . Thus

$$x = \log_a y \text{ if and only if } y = a^x$$

In other words, a log just tells us the power to which a base ( $a$ ) must be raised to get a particular number ( $y$ ).

**Example 9**  $u = \log_4 8$ . Here  $4^u = 8 \Leftrightarrow 2^{2u} = 2^3 \Leftrightarrow u = \frac{3}{2}$ .

### 8.1 Properties

1.  $\log_a (yz) = \log_a y + \log_a z$
2.  $\log_a (y/z) = \log_a y - \log_a z$
3.  $\log_a (1/z) = -\log_a z$
4.  $\log_a 1 = 0$
5.  $\log_a (y^b) = b \times \log_a y$
6.  $\log_a y = \log_a b \times \log_b y$  (Change-of-base formula)
7.  $a^{\log_a y} = y$

Why?

$$\begin{aligned} \log_a a^{\log_a y} &= \log_a y && \text{(Take logs on both sides)} \\ \log_a y \log_a a &= \log_a y && \text{(Property 5)} \\ \log_a y &= \log_a y && (\log_a a = 1) \\ \text{LHS} &= \text{RHS} \end{aligned}$$

Logarithms to base 10 are called *common* logarithms, and are denoted as  $\log_{10}$  or simply  $\log$ .

In Economics, we frequently make use of the *natural* logarithm which uses  $e$  as the base and is denoted  $\log_e$  or  $\ln$ . This is a bit unfortunate for you, the student, but the following rule should help you to remember.

If you see  $\ln x$ , then you know for sure that the base is  $e$ . If you see  $\log x$ , and the context is Economics (or math, statistics, physics and chemistry), then you can assume without incident that the base is  $e$ , unless the context explicitly tells you otherwise. Outside of Economics,  $\log x$  is assumed to be of base 10 (fields such as Engineering, and on your calculator) or base 2 (Computer Science, Information Theory).

**Example 10**

$$\begin{aligned}\ln e^3 &= 3 \ln e \\ &= 3 \log_e e \\ &= 3 \times 1 \\ &= 3\end{aligned}$$

**Example 11**

$$\begin{aligned}\ln \frac{e^3}{b} &= \ln e^3 - \ln b \\ &= 3 - \ln b\end{aligned}$$

**Example 12**

$$\begin{aligned}e^{\ln x} &= e^{\log_e x} \\ &= x\end{aligned}$$

**Example 13** *Suppose that a firm's output  $Y$  is related to capital input  $K$  and labour input  $L$  by the production function*

$$Y = \frac{1}{5} K^{0.4} L^{0.6}$$

*Express this as a linear relationship between the logarithms of the economic variables. We call this a log-linear relationship.*

*Let  $y = \log Y$ ,  $k = \log K$ ,  $\ell = \log L$ . Then*

$$\begin{aligned}\log Y &= \log \left( \frac{1}{5} K^{0.4} L^{0.6} \right) \\ \log Y &= \log \frac{1}{5} + \log K^{0.4} + \log L^{0.6} \\ \log Y &= \log 5^{-1} + 0.4 \log K + 0.6 \log L \\ \log Y &= -\log 5 + 0.4 \log K + 0.6 \log L \\ y &= 0.4k + 0.6\ell - \log 5\end{aligned}$$

The procedure of transforming a non-linear relationship into a linear one by taking logs is a useful one.

## 9 Solving Polynomial Equations

First let us distinguish between the terms *polynomial equation* and *polynomial function*. A polynomial function  $f(x)$  specifies a rule of mapping from  $x$  to  $f(x)$ . All values in the domain of the function satisfy  $f(x)$ . When we set the polynomial function  $f(x)$  equal to zero, the variable  $f(x)$  now disappears (having been assigned a zero value), the result is a polynomial equation in the single variable  $x$ . Only a select number of  $x$  values can satisfy  $f(x) = 0$  and qualify as its solution values. The solution  $x$  values are often referred to as the *roots* of the polynomial equation  $f(x) = 0$ .

### 9.1 The Quadratic Formula

In general, given a quadratic equation in the form

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

there are two roots, which can be obtained from the quadratic formula:

$$x_1^*, x_2^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

#### Example 14

$$2x^2 - 4x + 1 = 0 \quad (a \neq 0)$$

Here  $a = 2, b = -4, c = 1$ . So using the quadratic formula

$$\begin{aligned} x_1^*, x_2^* &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(1)}}{2(2)} \\ \Rightarrow x_1^* &= 1.71 \\ x_2^* &= 0.29 \end{aligned}$$

The quadratic formula is derived by means of a process known as “completing the square”. First, divide each term in the quadratic equation by  $a$  to get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Subtracting  $c/a$  from, and adding  $b^2/4a^2$  to, both sides of the equation, we get

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

The left side is now a “perfect square”, and thus the equation can be expressed as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

or, after taking the square root on both sides,

$$x + \frac{b}{2a} = \pm \frac{(b^2 - 4ac)^{1/2}}{2a}$$

Finally, by subtracting  $b/2a$  from both sides, the quadratic formula is obtained.

## 9.2 Factoring

The roots to higher order polynomial equations (such as cubic and quartic equations) are harder to find. One method is that of factoring the function.

**Example 15** *Solve the equation*

$$x^3 - x^2 - 4x + 4 = 0$$

*Evaluate the expression  $x^3 - x^2 - 4x + 4$  at  $x = 1$  :*

$$(1)^3 - (1)^2 - 4(1) + 4 = 0$$

*Thus, 1 is a root of  $x^3 - x^2 - 4x + 4$ . We could find the other roots by substituting in different values for  $x$  to find those that satisfy the equation. Using this method, we find that the expression  $x^3 - x^2 - 4x + 4$  can be written as the product of three factors  $(x - 1)$ ,  $(x + 2)$  and  $(x - 2)$ . Thus*

$$\begin{aligned} (x - 1)(x + 2)(x - 2) &= 0 \\ \Rightarrow x_1^* &= 1, x_2^* = -2, x_3^* = 2 \end{aligned}$$

Another approach to the above example is to divide  $x^3 - x^2 - 4x + 4$  through by  $(x - 1)$  and then factorise the resulting quadratic to get the other factors.

One way to divide a polynomial by a first degree polynomial is **synthetic division**. To use synthetic division, the divisor must be of the first degree and must have the form  $x - a$ . In general, if we divide a polynomial of degree  $n$  by a polynomial of degree 1, then the degree of the quotient will be  $n - 1$ . And the remainder will be a number.

**Example 16** *Use synthetic division to divide  $x^3 - 5x^2 + 3x - 7$  by  $x - 2$ .*

*1. Write the coefficients of the dividend:*

$$1 \quad -5 \quad +3 \quad -7$$

2. Put  $a$ , in this case 2, in a box to the right, leave a space, and draw a line:

$$\begin{array}{r} 1 \quad -5 \quad +3 \quad -7 \quad | \quad 2 \\ \hline \end{array}$$

3. Bring down the leading coefficient (1), multiply it with a (2), and write that product (1·2) in the second column:

$$\begin{array}{r} 1 \quad -5 \quad +3 \quad -7 \quad | \quad 2 \\ \quad +2 \\ \hline 1 \end{array}$$

4. Add:

$$\begin{array}{r} 1 \quad -5 \quad +3 \quad -7 \quad | \quad 2 \\ \quad +2 \\ \hline 1 \quad -3 \end{array}$$

5. Repeat the process.  $-3 \cdot 2 = -6$ . And so on, until all the coefficients have been exhausted.

$$\begin{array}{r} 1 \quad -5 \quad +3 \quad -7 \quad | \quad 2 \\ \quad +2 \quad -6 \quad -6 \\ \hline 1 \quad -3 \quad -3 \quad -13 \end{array}$$

The first three numbers, 1 - 3 - 3, are the coefficients of the quotient, and the final number, -13, is the remainder.

We have

$$\begin{aligned} x^3 - 5x^2 + 3x - 7 &= (x^2 - 3x - 3)(x - 2) - 13 \\ \text{Dividend} &= \text{Quotient} \cdot \text{Divisor} + \text{Remainder}. \end{aligned}$$

**Example 17** Use synthetic division to divide  $2x^5 + 3x^4 + 25x^2 - 1$  by  $x + 3$ .

There are a couple of points here. First, we must account for all six coefficients of the general form.

$$2 \quad +3 \quad +0 \quad +25 \quad +0 \quad -1$$

The coefficient of  $x^3$  is 0, as is the coefficient of  $x$ .

Next, the divisor is  $x + 3$ . But the divisor must have the form  $x - a$ .

$$x + 3 = x - (-3).$$

Therefore,  $a = -3$ .

Here is the synthetic division:

$$\begin{array}{r} 2 \quad +3 \quad +0 \quad +25 \quad +0 \quad -1 \quad | \quad -3 \\ \quad -6 \quad +9 \quad -27 \quad +6 \quad -18 \\ \hline 2 \quad -3 \quad +9 \quad -2 \quad +6 \quad -19 \end{array}$$

This tells us

$$2x^5 + 3x^4 + 25x^2 - 1 = (2x^4 - 3x^3 + 9x^2 - 2x + 6)(x + 3) - 19$$

So, returning to our earlier example:

**Example 18** Solve the equation

$$x^3 - x^2 - 4x + 4 = 0$$

We saw that  $(x - 1)$  was a factor of  $x^3 - x^2 - 4x + 4$ . Now, we can use synthetic division to divide  $x^3 - x^2 - 4x + 4$  by  $x - 1$ :

$$\begin{array}{r|rrrr} 1 & 1 & -1 & -4 & +4 \\ & & +1 & +0 & -4 \\ \hline & 1 & +0 & -4 & +0 \end{array}$$

Check that the remainder is 0 because we are dividing by a factor of the polynomial. Now we can write

$$\begin{aligned} x^3 - x^2 - 4x + 4 &= (x - 1)(x^2 - 4) \\ &= (x - 1)(x + 2)(x - 2) \end{aligned}$$

Thus,

$$\begin{aligned} (x - 1)(x + 2)(x - 2) &= 0 \\ \Rightarrow x_1^* &= 1, x_2^* = -2, x_3^* = 2 \end{aligned}$$

The next theorem assists in the search for roots.

**Theorem 1** Given the polynomial equation with integer coefficients

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

if there exists a rational root  $\frac{r}{s}$ , where  $r$  and  $s$  are integers without a common divisor except unity, then  $r$  is a divisor of  $a_0$ , and  $s$  is a divisor of  $a_n$ .

**Example 19** What are the roots of the equation

$$2x^4 + 5x^3 - 11x^2 - 20x + 12 = 0$$

With  $a_0 = 12$ , the only possible values for the numerator  $r$  in  $\frac{r}{s}$  are the set of divisors  $\{1, -1, 2, -2, 3, -3, 4, -4, 6, -6, 12, -12\}$ . And with  $a_n = 2$ , the only possible values for  $s$  in  $\frac{r}{s}$  are the set of divisors  $\{1, -1, 2, -2\}$ . Thus, we see that  $\frac{r}{s}$  can only assume the values

$$1, -1, \frac{1}{2}, -\frac{1}{2}, 2, -2, 3, -3, \frac{3}{2}, -\frac{3}{2}, 4, -4, 6, -6, 12, -12$$

Since we are solving a fourth degree polynomial, we can expect at most four roots and these turn out to be  $\frac{1}{2}, 2, -2, -3$ . Thus

$$\begin{aligned} 2x^4 + 5x^3 - 11x^2 - 20x + 12 &= 0 \\ \Rightarrow \left(x - \frac{1}{2}\right)(x - 2)(x + 2)(x + 3) &= 0 \end{aligned}$$

## 10 Solving Systems of Equations

A key concept in economics is that of equilibrium. In order for equilibrium to be obtained in a model, all variables in the model must simultaneously be at rest. When equilibrium has been reached, there is a lack of tendency to change. In other words, the system will not move away from this position, and even if there is some exogenous shock to the system, the system will automatically and instantaneously adjust back to the equilibrium position.

Because equilibrium is such an important concept in neoclassical economics, we spend a lot of time trying to solve economic models to determine what the equilibrium values of key variables (such as price and quantity) will be. However, there are 2 important conditions that must be met in order to be able to uniquely solve a system of equations:

1. Equations must be *consistent*. In other words, solving one equation in the model should not preclude the solution of another. For example:

$$\begin{aligned}x + y &= 9 \\x + y &= 8\end{aligned}$$

Clearly, these two equations are inconsistent, because if  $x$  and  $y$  add up to 8, they cannot also add up to 9! There is no solution to this system of equations.

2. Equations must be *functionally independent*. If two equations are functionally dependent, it means one equation can be derived from another. Thus, one equation becomes redundant and can be dropped from the system. For example:

$$\begin{aligned}2x + y &= 12 \\4x + 2y &= 24\end{aligned}$$

Clearly, the second equation is just two times the first equation. There are infinitely many solutions to this system of equations.

One way of finding a solution to an equation system is by successive elimination of variables and equations through *substitution*.

**Example 20** Consider the market model

$$\begin{aligned}Q_d &= Q_s \\Q_d &= a - bP && (a, b > 0) \\Q_s &= -c + dP && (c, d > 0)\end{aligned}$$

Using the equilibrium condition (the first equation) we can let  $Q = Q_d = Q_s$ , which gives us

$$\begin{aligned}Q &= a - bP \\Q &= -c + dP\end{aligned}$$



We can now substitute the first equation into the second and solve for  $P^*$

$$\begin{aligned} a - bP &= -c + dP \\ \Rightarrow (b + d)P &= a + c \\ \Rightarrow P^* &= \frac{a + c}{b + d} \end{aligned}$$

Note: While we have solved this part of the problem, it is not enough to stop there. Whenever you work on models, always try to think about the economic logic or intuition behind them.

Here we know that price can never be negative. This is ensured by the restrictions on the parameters:  $a, b, c, d > 0$ .

Now that we know the equilibrium value of  $P$ , we can solve for  $Q_d^*$  and  $Q_s^*$  via simple substitution.

$$\begin{aligned} Q_d^* &= a - bP^* = a - b \left( \frac{a + c}{b + d} \right) = \frac{a(b + d) - b(a + c)}{b + d} = \frac{ad - bc}{b + d} \\ Q_s^* &= -c + dP^* = -c + d \left( \frac{a + c}{b + d} \right) = \frac{-c(b + d) + d(a + c)}{b + d} = \frac{ad - bc}{b + d} \end{aligned}$$

Irrespective of which equation you substitute into, you get the same answer. This should not surprise you as we know that in equilibrium  $Q_d = Q_s$ .

**Example 21** Consider the market model

$$\begin{aligned} Q_d &= Q_s \\ Q_d &= 4 - P^2 \\ Q_s &= 4P - 1 \end{aligned}$$

As in the previous example, we can reduce this system of three equations to a single equation by substitution:

$$\begin{aligned} 4 - P^2 &= 4P - 1 \\ \Rightarrow P^2 + 4P - 5 &= 0 \end{aligned}$$

This is a quadratic equation which we can solve using the quadratic formula or by factoring. Here, the expression is easy to factor.

$$\begin{aligned} (P + 5)(P - 1) &= 0 \\ \Rightarrow P_1^* &= 1 \\ P_2^* &= -5 \end{aligned}$$

However, we know that prices can never be negative, so in this case, the only economically admissible solution is that  $P^* = 1$ .

Substituting  $P^* = 1$  into the demand and supply functions gives us  $Q^* = Q_d^* = Q_s^* = 3$ .