ECO 4112F	Chapter 9.2-10.1

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Lagrange fo of the Remainder

We can express as

$$R_n = \frac{\phi^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1} \qquad (9.15)$$

here p is a number (where we cant ito evaluate $\phi(x)$), & x_0 (the pt around hich we expand ϕ)

This ooks ke the (n+1)st term n than Taylor expansion

Can we calculate R_n?

No, we dont know x.

Did I tell you why?

ut w will now for the ... = 0, ... constant ... approximation to $\phi(x)$

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know $\quad n=0 \quad\quad n=$

$$\phi(x) = \phi(x_0) + \phi'(x_0)(x-x_0) + \phi''(x_0)(x-x_0)^2 + \ldots + \phi^{(n)}(x_0)(x-x_0) + R_n$$

s

$n=0$ (

$$\phi(x) = P_n + R_n = P_0 + R_0$$

$$\phi(x) = \underbrace{\phi(x_0)}_{P_0} + \underbrace{\phi'(p)(x-x_0)}_{\text{from formula 91 above}}$$

$$\therefore \underbrace{\phi(x) - \phi(x_0)}_{} = \phi'(p)(x-x_0)$$

s difference can be expressed as the product of $\phi'(p)$, & the difference btwn x & x_0.

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See ◁

on page 249

x_0 = pt expain n

x = where we want to evaluate $\phi(x)$ is

try to approximate $\phi(x)$

◁ B) with $\phi(x_0)$ (dist x_0)

$\phi(x) - \phi(x_0)$

$= CB$

~~Pick a pt~~ D, ~~btwn x & x_0~~

Pick a pt D on curve, ~~between~~ A & B, st tangent line ~~pt p~~) is parallel to AB. (we can be can assumed $\phi(x)$ is continuous & has cont. derivs)

$\therefore R_0 = CB = \frac{CB}{AC} AC = \begin{pmatrix}\text{slope}\\ AB\end{pmatrix}$ A

(slope tangent at p) by A

$\phi'(p)(x - x_0)$

I've shown you the formula

$$R_n = \frac{\phi^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+}$$

pscusse (p) ϕ o

works for $n = 0$

So "d will $R_n \to 0$ as n

mm

p'd

so, $P_n \to \phi(x)$ as $n \to \infty$

We say the Taylor series converges to $\phi(x)$ at $x = x_0$ &

we can write $\phi($

$$x) = \frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)(x - x_0)}{1!} + \frac{\phi''(x\ x_0)(x - x_0)^2}{2!}$$

If

i ∞ a big enough

be accurate an a rox

(x) as we want

in 10.2)

$$\left(\frac{\quad}{m} \right)^m$$

That was nice, & there another way to approx e & show $e = 2\ 828$

Taylor Series!

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- How s e defined?
- Why do we like $y = e^x$? Does it have economic meaning

Consider $f(m) = \left(1 + \frac{1}{m}\right)^m$

What happens as m

$f(1) = 2 = \left(1 + \frac{1}{1}\right)^1$

$f(2) = 2.25 = \left(1 + \frac{1}{2}\right)$

$f(3) = 2.37037\ldots$

etc etc

As $m \to \infty$, $f(m)$

$\therefore e = m +$

(2)

Use usual formula

We know $= e^x$

$= e^x = \;\; = y'' = \ldots = y^{(n)}$

for $\phi(x) = e^x$

$\phi(x)$ evaluated at $x = c$ (Mac laurin

$$\phi(x) = \phi(0) + \phi'(0)x + \frac{\phi''(0)x^2}{2!}$$

$$\frac{\phi^{(n)}(0)x^n}{n!} \quad + R_n$$

$$\phi(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n} + R_n$$

What is R_n?

$$R_n = \frac{\phi^{(n+1)}(p)\,x^{n+1}}{(n+1)!} = \frac{e}{(n+1)!}x^{+}$$

Remember $\phi^{(n+1)}(x) = e^x \;\therefore\; \phi^{(n+1)}(p)$ p

Luckily we know $(n+1)!$

faster than x^{n+1} as $n \to \infty$

Why? Basic Math Fact (you don't need to prove)

③

$$= \frac{e^{p} x^{n+}}{+1)} \to 0^{\circ} \text{ as } n \to \alpha$$

ember e^{p} is a constant

n Series converges

m $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$

$\therefore e^{c} = e = +1 + \frac{1}{2!} + \frac{1}{3} + \frac{}{4} + \ldots$

$= 2.718289$

For 5 dec places te $e = 2.7182$

Economic nterpretation of e

We know $e^{1} = \lim_{m \to \alpha} \left(+ \frac{1}{m+} \right)^{1} = x^{2}$

Do this have ~~economic~~ economic meaning?

(wrong)

Suppose I give you a 1\$ & offer you 100% per annum (year)

If we compound interest 1X a
The value of our asset after
1 year is 2\$ denote as V(
$V(m)$ = Value of \$1 if we compound
m times a year, $V(m)$ =
value after 1 year.

$V(1)$ initial principal(1+ int rate)
$= 1(1+100\%) = 1(1+\frac{1}{1})^1 = 2\$$

<u>Semi - Annually</u> $(m=2)$

$V(2) = ?$ After 6 months, we
have \$1 50 (100% a year 50%
semi annually)
\$1 50 earns 50% again
After year, we have $1.50(1+\frac{1}{2})$

$V(2)$ $(1+50\%)(+0\%)$ $(1+\frac{1}{2})$

$V(3)$ $(1+\frac{1}{3})^3$ $V(4)$ $(1 \quad \frac{1}{4})^4$

$$V(m) = \left(1+\frac{1}{m}\right)^m$$

value of \$1, after 1 year, (100% a year)
compounding m x a year

(S)

W. If we compound continu

$\therefore \lim_{m \to \infty} r(m) \quad \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m$

Nice

$\frac{1}{1}$

Take 1\$ with int rate = 100% year compound continuously, you get $r(m)$ as $m \to \infty$ =

NB 100% = nominal int rate

Effective int rate = approx 172%

Why? 1\$ grows to e\$ = R2.718

$\therefore \quad r = \frac{\Delta amt}{amt} = \frac{1.71828}{R1} \quad r = 172$

$= .72 = .7$

What if we start with A\$ not 1\$, & compound with annual int rate = r, not 100%, and for t years, not year

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if 1\$ grows to e\$ after 1 year, it will grow to e^2 after 2 years ($e \cdot e = e^2$ = every 1\$ grows to e\$, $\therefore$ e grows to e e).

After 3 years e^3

After t years, e^t

Change 1\$ to A\$

1 grows to e (1 year)

A grows to Ae

1 grows to e^t (t years)

A grows to e^t (t years)

What if $r = 0.05$ (ie 5 percent)

Ae^t will become Ae^{rt}

Why? V(m) becomes $= A\left(1+\frac{r}{m}\right)^{mt}$

$\boxed{\text{Check}}$ $= Ae^{rt}$

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Are we okay with this?

Previously $V(m) = \left(1 + \frac{1}{m}\right)^m$

where was $r = 1$ (100%)?

$$= \left(1 + \frac{r}{m}\right)^m$$

emember if my annual int rate is r (eg $r = 1 = 100\%$) but I compound m times per year (eg $m = 3$) then actually the int rate in each period is $\frac{r}{m}$ (eg $\frac{1}{3}$)

So: $V(m) = A\left(1 + \frac{r}{m}\right)^{mt} = Ae^{rt}$??

Because $e = \left(1 + \frac{1}{m}\right)^m = \left(1 + \frac{r}{m}\right)^m$ where int rate $= r$

Int is compounded m times a year in year ∴ mt times in t years.

(8)

We can rewrite $v(m)$

$$A\left(1+\frac{r}{m}\right)^{mt} = A \quad rt \text{ (why?)}$$

$$= A\left[\left(1+\frac{r}{m}\right)^{m/r}\right]^{rt}$$

$$A\left[\left(1+\frac{}{w}\right)^{w}\right]^{rt} \quad w \leftarrow w$$

$m \to \infty \quad = \quad A \quad = A \therefore$

$w \to \infty$

$$v(m) \to A(e)^{rt}$$

$\frac{1}{w} = m \quad \lim_{\infty} v(m) \quad A^{rt} \quad$ As

<u>Remember</u> t = discrete =

<u>Given</u> m = num times we compound per year

$\therefore$ t must $k\left(\frac{1}{m}\right)$, where $k = 1, 2, 3.$

eg m 4, we can have no (^

periods $= \frac{1}{4}$ year $\frac{1}{2}$ year, $\frac{3}{4}$ year)

etc

ie we compound once $(t=\frac{1}{4})$, twice $(t=\frac{1}{2})$ three times $(t=\frac{3}{4})$, etc etc

As $m \to \infty$, $\frac{1}{m} \to 0$, $t = k(\frac{1}{m})$

$\Rightarrow$ $t \to$ continuous (can take on any number)

Instantaneous Rate of Growth

See table 10.1 p264

Make sure you understand & can replicate

We can interpret Ae^{rt} in other ways.

= exponential growth

(eg of popn, wealth, capital etc)

can be interpreted as the nstaneous growth of rate

Why?

$\frac{dV}{dt}$

rate of growth

of V

$\boxed{NB}$

V = stock, value at a pt t

ΔV flow, we must know what period of time t is involved

= inst rate of growth

= constant all all pts of 'time'

= we need to know the time period to be able to interpret r

r = constant, but V in ^

= more each time

N = ... creates ^ & we

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$-r e^{rt} = r$

eg $y = 23e^{0 \cdot t}$ rate of growth is 4% per perio.

Continuous vs Discrete Growth

- Interesting, but economic relevance?
- Can we use Ae^{rt} for ~~con~~ discrete growth? Yes.
- We can convert discrete to continuous.

Given discrete compounding:

$A, A(1+i), A(1+i)^2, A(1+i)^3, \ldots$

i = effective int rate per period

exponent = num of periods

Can write $(1+i) = b$

$\therefore$ $A, Ab, Ab^2, Ab^3, \ldots$ ie Ab^t

$t = 1, 2, 3,$ (integer because this is discrete)

$(1+i)^t = A$ ∴

$> 0 \; (1+i)^{-t} \; = 0 < 0$

∴ there must exist r s

= ... (because $e > 0$)

$A(1+i)^t = A(e^r)^t$

$= Ae^{rt}$

for any t, Ae^{rt} gives same s

$A(1+i)^t$

∴ can use co ers on ever

if discrete

Discounting & Negative q

hat if given FV, want to find PV?

Now A is

V is $q = i + 1$

∴ Discrete

V' $1 + 1 = q$

e^{-rt} (ctor

$-r$ nstantaneous rate of growth

rate of decay.

Dscounting = negative growth