1

Chapter 9

P220 – P254

Please Revise

2nd derivatives

Previously, non goal EQM

EQM from balancing of different forces in the economy.

Now: goal EQM

EQM = optimum state for an economic unit (HH, firm, economy)

Optimum & Extreme Values

↳ we want to know how to maximise or minimise a certain variable – ie find extrema, & choose the "best" of them

eg $\max\limits_{x} \pi(Q) = R(Q) - C(Q)$

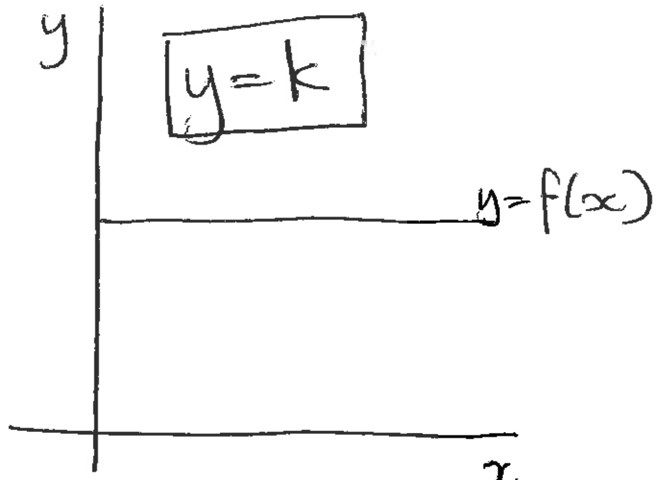
$\pi(Q)$ = objective fn, Q = choice var.

(2)

We have to choose the q which maximises Π

So given $y = f(x)$

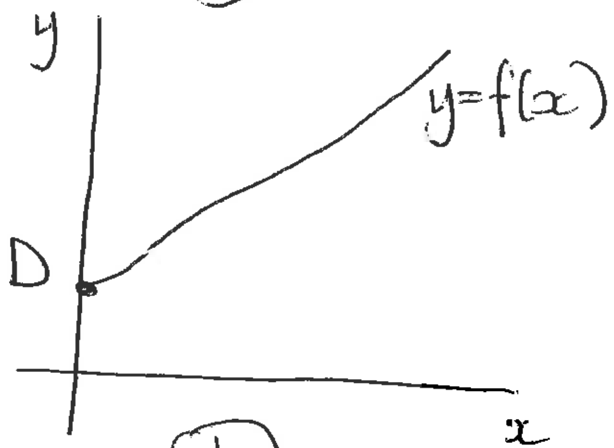
Assume $f(x)$ is continuously differentiable.



(a)

- No value of x yields a different value of y

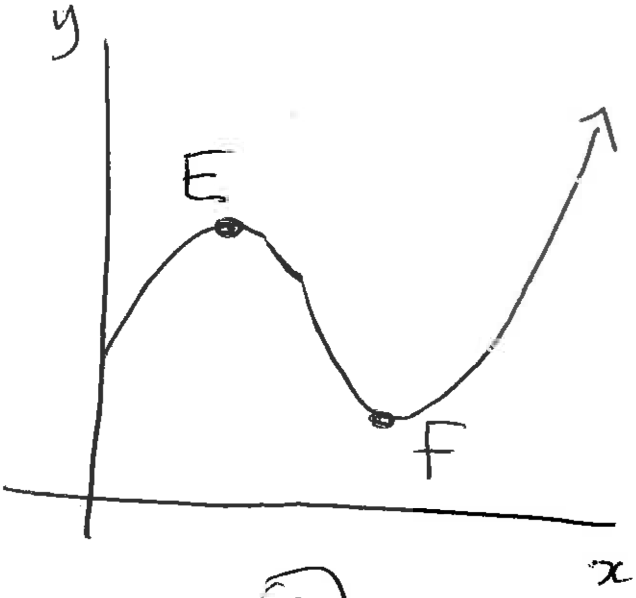
$\therefore$ any pt on y is a max. or min.



(b)

- $f'(x) > 0$.
- pt D is a minimum.
- there is no finite max.

(3)



- E, F are relative / local extrema.
- F = local min
  E = local max

(c)

- A fn can have many relative extrema, either maxima or minima.
- Generally LH end pt is of no interest to us. Why?
- How to find absolute min/max?

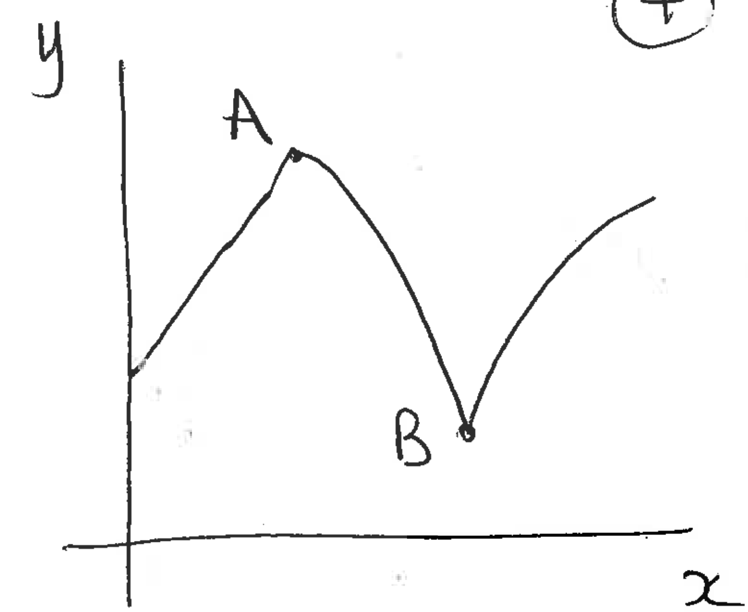
Find all local min, pick smallest
max, pick biggest

- We know

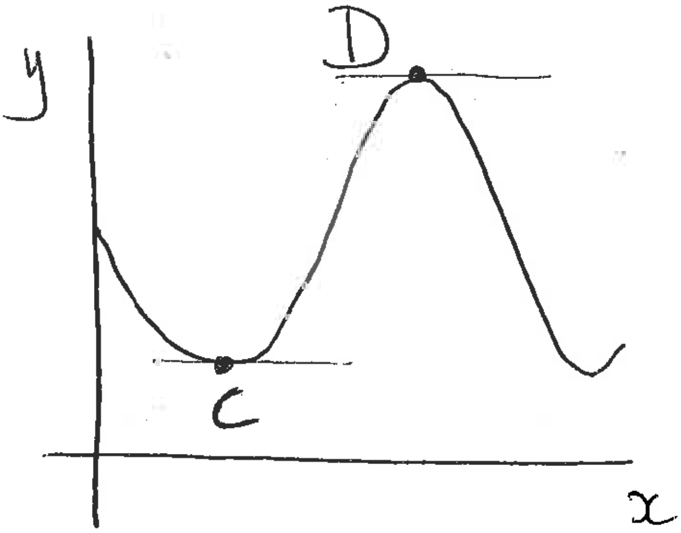
if $y = f(x)$, & an extremum exists at $x = x_0$, then either

(1) $f'(x_0)$ doesnt exist or (2) $f'(x_0) = 0$

(4)



- At A, B we have local extrama, but $f'(x)$ does not exist at A or B. Why? (*)



- Here at C & D $f'(x) = 0$
- We assume a continuously differentiable fn, hence wont encounter A, B

(*) At any sharp pt on a fn, the fn is not differentiable at that pt, ie $f'(x)$ does not exist, because $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

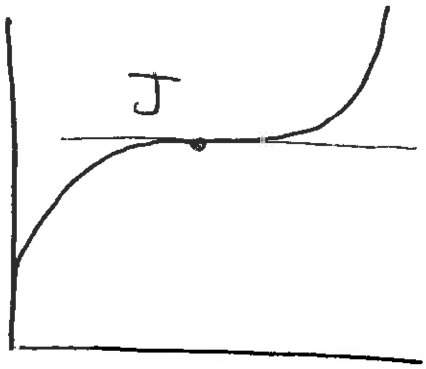
& here the LH lim $\neq$ RH limit, hence the limit does not exist

(P146)

(S)

- $f'(x) = 0$ is a <u>necessary</u> condition for a relative extremum — but <u>not sufficient</u>

- <u>eg</u>



$f'(x) = 0$ at J, but J is not an extremum pt. (*)

<u>1st Derivative Test</u>

If $f'(x_0) = 0$, then $f(x_0) =$

(a) a relative max if $f'(x)$ is positive on left of x_0, & negative on the right

(b) a relative minimum if $f'(x)$ is -ve on LHS of x_0, & +ve on RHS

(c) neither if $f'(x)$ doesnt change sign on either side of x_0.

(*) J = inflection pt.

(6)

x_0 = critical value (if $f'(x_0)=0$)

$f(x_0)$ = stationary value

$(x_0, f(x_0))$ = stationary pt.

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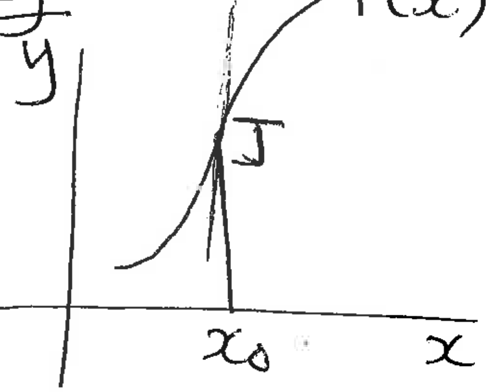
if $f'(x_0)=0$ (necessary)

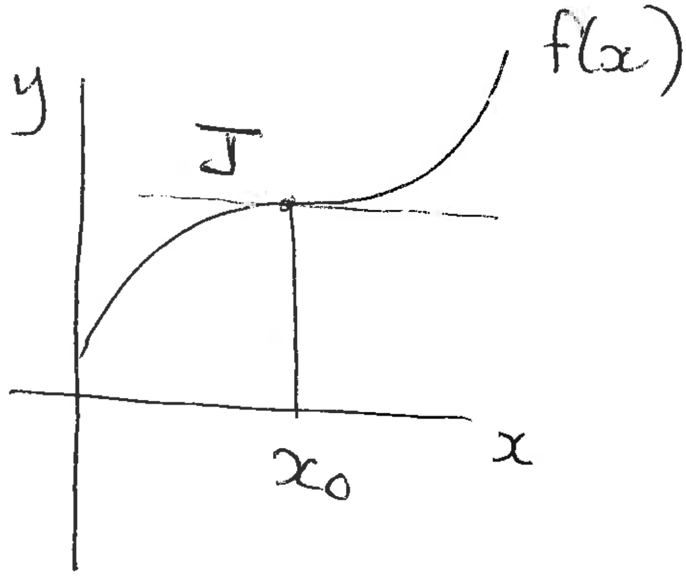
& either (a) or (b) holds, then we have a relative min or max.

B.T.W

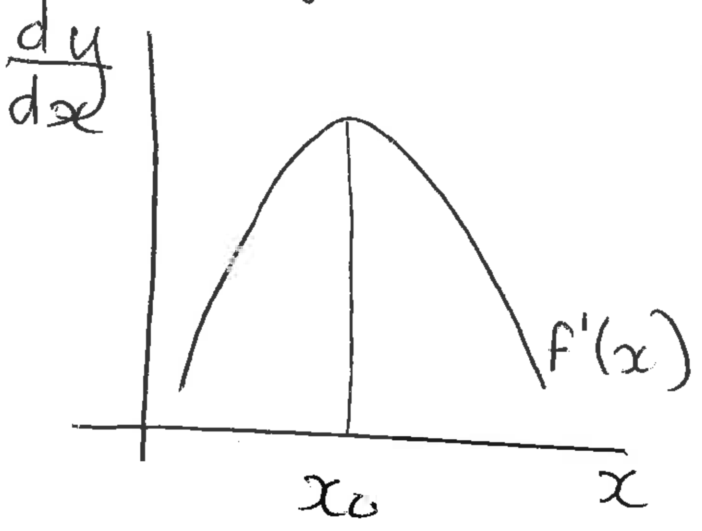
Inflection pt = pt where $f'(x)$ (not $f(x)$) reaches an extreme value

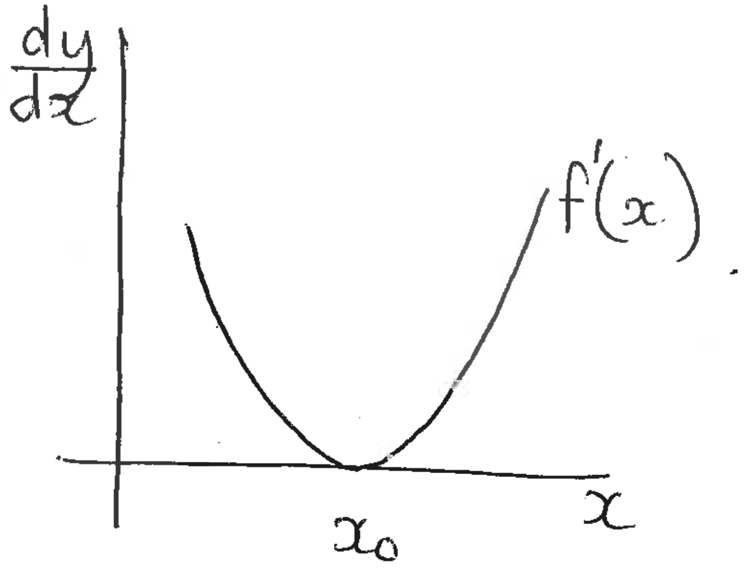
eg





We graph $f'(x)$ for these cases:





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eg Given $y = f(x) = x^3 - 12x^2 + 36x + 8$

$$f'(x) = 3x^2 - 24x + 36 = 0$$

$$3(x^2 - 8x + 12) = 0$$

$$3(x-6)(x-2) = 0$$

$\therefore$ $x = 6$ $\quad$ $x = 2$

$\therefore$ $y = 8$ $\quad$ $y = 40$

A $\quad$ B

What kind of stationary pts are these?

Check $f'(x)$ on either side.

eg $f'(0) = 36 > 0$
$f'(3) = -9 < 0$

pt B is a max (relative)
pt A is a min (similarly
check)

See graph P 226

- 1st derivative test helps to locate & classify extrema.
- ex 2 P 226 – HW.

2nd Order Derivatives

- $y = f(x)$
- $y' = f'(x) = \frac{dy}{dx}$
- $y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

– if f'' exists, then f is twice differentiable
– if continuous, f is twice continuously differentiable

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Notation: $f \in C^{(2)}$ / $f \in C''$

We can find:

$f'''(x), f^{(4)}(x), \ldots f^{(n)}(x)$

OR $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots \frac{d^ny}{dx^n}$

We assume our fns have continuous derivatives to the order we need.

Revision – P228, ex 1, 2.

eg
$$y = x^4 + 3x^2 + 2$$
$$y' = 4x^3 + 6x$$
$$y'' = 12x^2 + 6$$
$$y''' = 24x$$
$$\frac{d^4y}{dx^4} = 24$$

- $f^{(5)}(x) = 0$, as are $f^{(6)}(x), f^{(7)}(x)$ etc

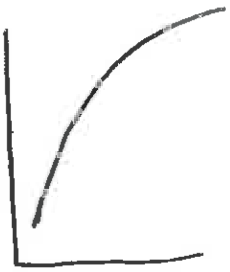
$f'' =$ rate of change of f'

$f' > 0$ ⇒ y ⇒ is increasing

$f' < 0$ ⇒ y ⇒ is decreasing

$f'' > 0$ ⇒ slope of y ⇒ is increasing

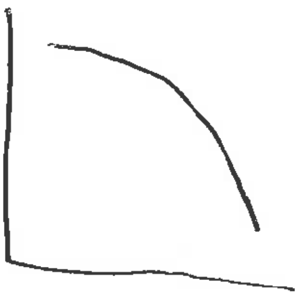
$f'' < 0$ ⇒ slope of y ⇒ is decreasing



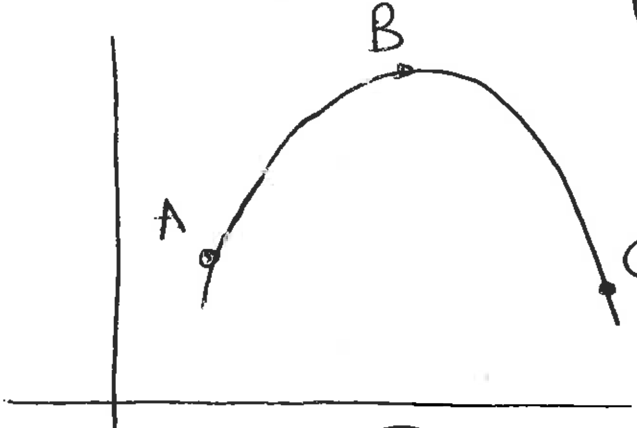
$f' > 0$ $\quad$ $f'' < 0$

$f' > 0$ $\quad$ $f'' > 0$

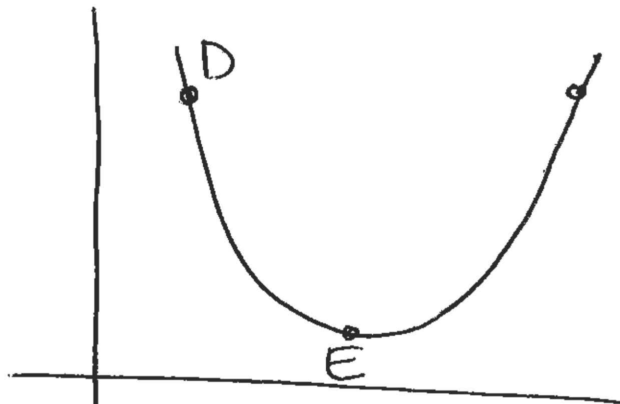
$f' < 0$ $\quad$ $\cancel{f'' < 0}$ $f'' > 0$



$f' < 0$ $\quad$ $f'' < 0$



(a)



(b)

- here f' goes from +ve to 0 to −ve
  ∴ $f'' < 0$

- f' goes from −ve to 0 to +ve
  ∴ $f'' > 0$

(a) strictly convex

(b) strictly concave

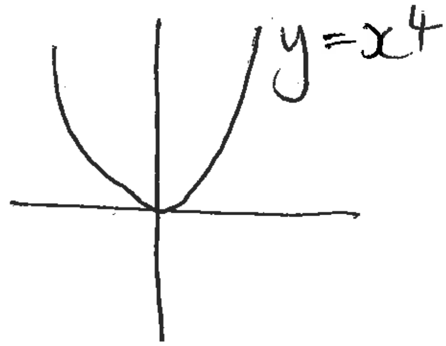
- If we pick any 2 pts M & N on the curve, the line MN must entirely below the curve = STRICTLY concave
  above = convex

If MN touches the curve, then f is only concave/convex, not strictly

- If $f''(x) < 0 \ \forall x$, then f is strictly concave

  $f'' > 0 \ \forall x \Rightarrow$ strictly convex

- Does <u>strict</u> convexity / concavity $\Rightarrow$ $f''(x) > 0$ / $f''(x) < 0$ for all x? NO



eg $y = x^4$

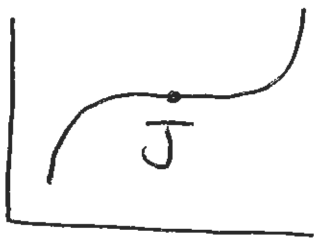
$y' = 4x^3$

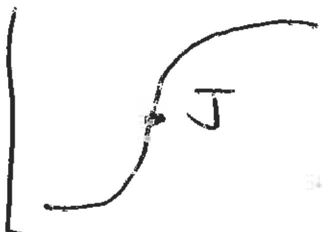
$y'' = 12x^2$

$y'' = 0$ if $x = 0$, $\therefore$ $f''(x) \geq 0$

but a chord MN lies entirely above the chord.

Inflection pts





- At J, $f(x)$ changes concavity to convexity or vice versa;
- f' reaches an extreme value

eg $y = ax^2 + bx + c$

is y concave?

$y'' = 2a$

if $a > 0$ y is strictly convex, etc

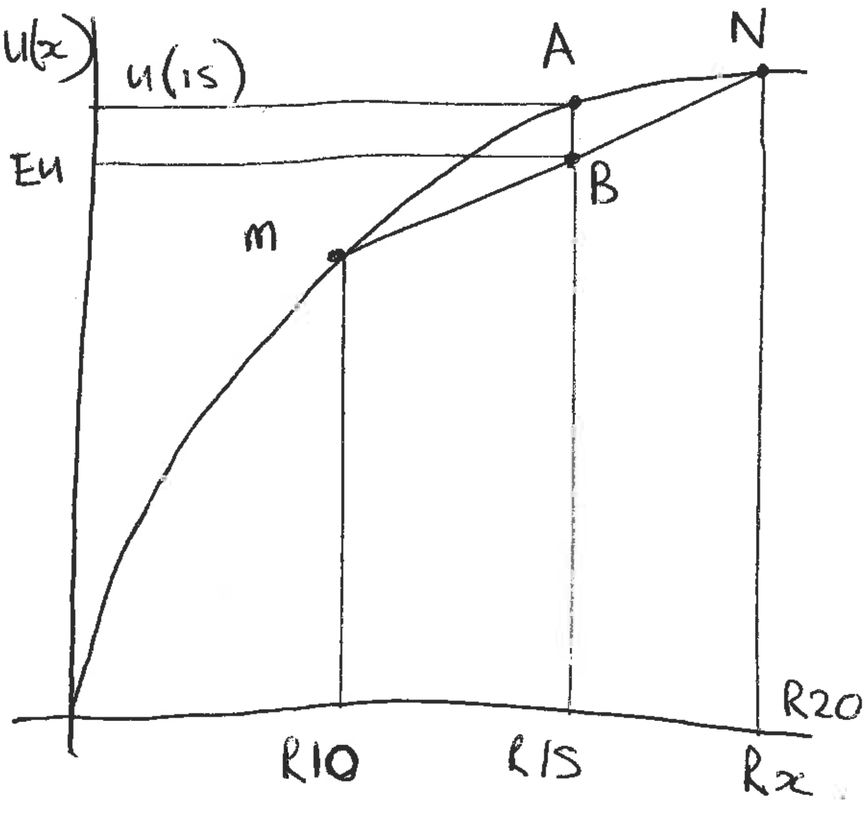
Risk Profiles & Concavity P231

You have R15, bet to either lose R5 or gain R5 with equal probability.

$E(\text{bet}) = 0.5(5) + 0.5(-5) = 0$

∴ fair bet.

Do you take the bet?

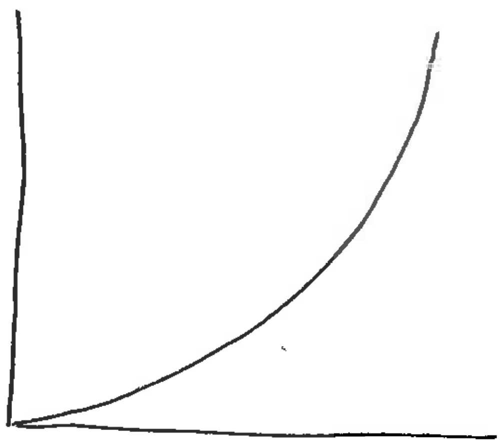


- dont take bet, get U at A
- take bet, expected value = R15, expected U is EU, from B
- MN below U(x), strictly concave

- Do you take the bet? No

EU from playing < U from not playing

$\Rightarrow$ risk averse



P232

HW Complete graph

risk loving

= strictly convex

- What is $U(x) = kx$?

Risk Neutral

2nd Derivative Test

for relative extremum:

If $f'(x_0) = 0$, then $f(x_0)$ is:

① a relative max if $f''(x_0) < 0$

② a relative min if $f''(x_0) > 0$

– Is this better than 1st derivative test?

What if $f''(x_0) = 0$?

L we have either relative max/min or inflection pt.

L then use 1st derivative or anothe test.

<u>EG</u> $y = g(x) = x^3 - 3x^2 + 2$

$g'(x) = 3x^2 - 6x$

$g''(x) = 6x - 6$

$g'(x) = 0 = 3x(x-2)$

$x = 0$, $x = 2$

$g''(0) = -6 < 0$ $\quad$ $g''(2) = 6 > 0$

relative max at $x = 0$ $\quad$ relative min at $x = 2$.

$f'(x) = 0$ is <u>necessary</u> in 2nd derivative test : called the

FOC : 1st order condition

If FOC fulfilled, $f''<0$ or $f''>0$ is sufficient for a max or min
↳ these are SOC

FOC — necessary but not sufficient
SOC — sufficient but not necessary
1st reason — inflection pts
2nd reason — $f''(x_0)=0$

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ie if $(x_0, f(x_0))$ is a minimum this implies $f'(x_0)=0$, but does not always imply $f''(x_0)>0$

2nd order necessary conditions:

$$\begin{matrix} \text{max} \\ \text{min} \end{matrix} \Rightarrow \begin{matrix} f''(x_0) \leq 0 \\ f''(x_0) \geq 0 \end{matrix}$$

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eg: <u>Profit Maximisation</u>

$$\pi(Q) = R(Q) - C(Q)$$

Where is π maximised?

<u>FOC</u> : $\pi'(Q) = 0$

$$\frac{d\pi}{dQ} = R'(Q) - C'(Q) = 0$$

$$\therefore \quad R'(Q) = C'(Q)$$

ie $MR = MC$

Is this definitely a max?

Check <u>SOC</u>

$$\frac{d^2\pi}{dQ^2} = R''(Q) - C''(Q) \leq 0$$

iff $R''(Q) \leq C''(Q)$

- This is the necessary condition – the sufficient is $\pi'' < 0$
- So we need slope MR < slope MC

See P237

- We see FOCs are met at Q_1 & Q_3 : $\pi' = 0$
- SOC slope MR < slope MC

at Q_1, both have negative slopes eg MR slope = −5
MC " = −8 (steeper)

∴ SOC not met

at Q_3 slope MC +ve
slope MR −ve

SOCs are met

- Given a specific functional form, we could just use simpler SOC $\pi'' < 0$
- HW p238 eg 3

Cubic Cost fn

- $C(Q) = aQ^3 + bQ^2 + cQ + d$

- Generally cost fns have segments with decreasing MC (ie concave) & increasing MC (convex).
- The cost function needs to be upward sloping everywhere. Why?

- ie we need $MC > 0$ for all Q

This is possible if the absolute minimum of MC is positive.

$\therefore \quad MC = C'(Q) = 3aQ^2 + 2bQ + c$

This is a parabola: $\cup$ or $\cap$

We need $\cup$ shape

ie $a > 0$

We also need $\min MC > 0$

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Where is min of mc?

FOC: set $mc' = 0$

$$mc' = 6aQ + 2b = 0$$

$$Q^* = \frac{-b}{3a}$$

Is $mc(Q^*)$ a min or max?

Check SOC: $mc'' = 6a$

$mc'' > 0$ if $a > 0$.

$\therefore$ at Q^*, mc is minimised.

We rule out $Q \leq 0$

$\therefore$ $Q^* = \frac{-b}{3a} \Rightarrow b < 0$ why?

So what is $mc(Q^*)$ & is it positive?

$$mc(Q^*) = 3a\left(\frac{-b}{3a}\right)^2 + 2b\left(\frac{-b}{3a}\right) + c$$

$$= \frac{3ac - b^2}{3a}$$

$\therefore$ $mc(Q^*) > 0$ if $3ac - b^2 > 0$

ie $b^2 < 3ac$

This implies $c > 0$. Why?

What about d?

$TC = c(Q)$

$c(0) = d$ ie y intercept

If $Q = 0$, $TC = d$ is fixed cost

$\therefore d > 0$

So: $a, c, d > 0$; $b < 0$; $b^2 < 3ac$

These ensure that TC is upward sloping over all $Q \geq 0$

ie $MC > 0$

Under imperfect competition, MR is downward sloping.

Why?

How do we check MR is always downward sloping.

Given $AR = f(Q)$

$$AR = f(Q) = \frac{TR}{Q}$$

$$\therefore TR = Qf(Q)$$

$$\therefore MR = f(Q) + Qf'(Q)$$

What is slope of MR?

$$\frac{d}{dQ}(MR) = f'(Q) + f'(Q) + Qf''(Q)$$

$$= 2f'(Q) + Qf''(Q)$$

If MR slopes down, $MR' < 0$

$\therefore$ If AR slopes down, then $f'(Q) < 0$

what is $Qf''(Q)$?

If AR is strictly convex, $f''(Q) > 0$

& it is possible that

$$MR' = 2f'(Q) + Qf''(Q) > 0$$

See ex 4 P240 = $\boxed{HW}$

Maclaurin & Taylor Series

What $f'(Q_0) = 0$, ie f is an extremum at $Q = Q_0$, but $f''(Q_0) =$

∟ we cant classify the extremum

∟ need the n derivative test

1st, find out how to expand a fn $y = f(x)$ into a series of terms, either around the pt $x = 0$, or $x = x_0$

Maclaurin — Taylor

- We want to transform a fn, eg $y = \frac{1}{1+x}$, $y = e^x$, $y = x^3 + x^2$, into an nth degree polynomial, ie $y = a + bx + cx^2 + dx^3 + \ldots$

- a, b, c, d etc will expressed in terms of $y'(x_0)$, $y''(x_0)$ etc etc etc

∟ power series

= sum of power fns

Maclaurin

eg $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n$

Express $y = f(x)$ as a series of terms with $(a_0, a_1, a_2 \ldots)$ expressed i.t.o $f(0), f'(0), f''(0)$ etc.

What are f', f'' etc?

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots + n a_n x^{n-1}$

$f''(x) = 2a_2 + 3(2)a_3 x + 4(3)a_4 x^2 + \ldots + (n)(n-1)a_n x^{n-2}$

$f'''(x) = 3(2)a_3 + 4(3)(2)a_4 x + 5(4)(3)a_5 x^2 + \ldots + n(n-1)(n-2)a_n x^{n-}$

$f^4(x) = 4(3)(2)a_4 + 5(4)(3)(2)a_5 x + \ldots + n(n-1)(n-2)(n-3)a_n x^{n-4}$

$\vdots$

$f^n(x) = n(n-1)(n-2)(n-3) \ldots (3)(2)(1)a_n$

We differentiated n times, left with 1 constant.

Let's evaluate these derivatives when $x = 0$.

$\therefore$ $f'(0) = a_1$

$f''(0) = 2a_2$

$f'''(0) = 3(2)a_3$

$f^4(0) = 4(3)(2)a_4$

$\therefore$ $f^5(0) = 5(4)(3)(2)a_5$

$\therefore$ $f^n(0) = n(n-1)(n-2)\dots(3)(2)(1)a_n$

$n! = n$ factorial

$= n(n-1)(n-2)\dots(3)(2)(1)$

$\therefore$ $f'(0) = 1!\,a_1$

$f''(0) = 2!\,a_2$

$f'''(0) = 3!\,a_3$

$f^4(0) = 4!\,a_4$

$\therefore$ $a_1 = \dfrac{f'(0)}{1!}$

$a_2 = \dfrac{f''(0)}{2!}$

$a_3 = \dfrac{f'''(0)}{3!}$

etc.

$\therefore \quad a_n = \dfrac{f^n(0)}{n!} \qquad$ NB $\quad a_0 = f(0)$

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^n(0)}{n!}x^n$$

This is the Maclaurin Series (when $x_0 = 0$)

- It is the expansion of $f(x)$ around $x = 0$.
- We sub $x = 0$ into all derivative

$$f(x) = \sum_{i=0}^{n} \frac{x^i \cdot f^i(0)}{i!} = \sum_{i=0}^{n} a_i x^i$$

So we found derivatives of $f(x)$, evaluated them at $x = 0$, used this to solve for the a_i, & subbed the a_i back to the original $y = f(x)$

- We solved for the a_i i.t.o the derivatives
- We get a different fn $f(x)$, equivalent to the original.

eg $y = 2 + 4x + 3x^2$

$y' = f'(\ ) = 4 + 6x$

$y'' = f''(x) = 6$

$y''' = y^n = 0$

$\therefore f'(0) = 4$

$f''(0) = 6$

$\therefore f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

$= 2 + 4x + \frac{6}{2}x^2$

$=$ orignal f

So: $f(x) = \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f^3(0)x^3}{3!} + \frac{f^4(0)x^4}{4!} + \ldots + \frac{f^n(0)x^n}{n!}$

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We can expand around $x = x_0$

Taylor Series

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0)$$

$$+ \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^n(x_0)(x - x_0)}{n!}$$

Why?

lets interpret any x as deviation from x_0 ie $x = x_0 + \delta$ where δ = deviation

We had: $f(x) = 2 + 4x + 3x^2$

$\therefore$ Now $f(x) = 2 + 4(x_0 + \delta) + 3(x_0 + \delta)^2$

$$f'(x) = 4 + 6(x_0 + \delta)$$

$$f''(x) = 6$$

x_0 is given & here only δ is variable $\therefore$ $f(x)$ here $= g(\delta)$

$$g(\delta) = f(x) = 2 + 4(x_0 + \delta) + 3(x_0 + \delta)^2$$

$$g'(\delta) = f'(x)$$

$$g''(\delta) = f''(x)$$

Let's expand $g(\delta)$ around zero, ie find Maclaurin Series

$\delta = 0$

We know $g(\delta) = \dfrac{g(0)}{0!} + \dfrac{g'(0)\delta}{1!} + \dfrac{g''(0)\delta^2}{2!}$

BTW: Why dont we add more terms? Because $g'''(\delta) = 0$ & so are all following deriv's.

If $\delta = 0$, this implies $x = x_0$

& we know $g(\delta) = f(x)$ ∴ for $\delta = 0$

$$g(0) = f(x_0)$$
$$g'(0) = f'(x_0)$$
$$g''(0) = f''(x_0)$$

∴ Sub in

$$f(x) = g(\delta) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

Taylor Series: expansion of $f(x)$ around a pt x_0

So for $f(x) = 2 + 4x + 3x^2$

We have $f(x_0) = 2 + 4x_0 + 3x_0^2$

$f'(x_0) = 4 + 3(2)x_0$

$f''(x_0) = 6$

$\therefore$ Taylor Polynomial is

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)(x-x_0}{2!}$$

$$= \underbrace{2 + 4x_0 + 3x_0^2} + \underbrace{(4 + 6x_0)(x - x_0)}$$

$$+ \frac{6}{2} \cdot (x - x_0)^2$$

$$= 2 + 4x + 3x^2$$

$\therefore$ The Taylor poly, as with the Maclaurin fn, represents $f(x)$ correctly.

Maclaurin

$$\sum_{i=0}^{n} \frac{f^{i}(0) \cdot x^{i}}{i!}$$

Taylor

$$\sum_{i=0}^{n} \frac{f^{i}(x_0)(x - x_0)^{i}}{i!}$$

Maclaurin is Taylor with $x_0 = 0$.

Given a fn $f(x)$, & we would like to know the value of $f(7)$.
You can sub into the fn $f(x)$

OR

you can pick x_0 to be any number, sub in $x = 7$, & the answer from Taylor Series will equal to $f(7)$

eg

for $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

if $x_0 = 3$

We have

$$f(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \dots + \frac{f^{(n)}(3)}{n!}(x-3)^n$$

$f(x)$ now as any arbitrary fn.
Is it the same?

For any $\phi(x)$, we can express as a nth degree poly, if $\phi(x)$ has finite, continuous derivatives at $x = x_0$.

If we know $\phi(x_0)$, $\phi'(x_0)$, $\phi''(x_0)$ etc

Then
$$\phi(x) = \left[\frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)}{1!}(x-x_0) + \frac{\phi''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{\phi^n(x_0)}{n!}(x-x_0)^n\right] + R_n$$

$$\phi(x) = P_n + R_n$$

P_n = nth degree poly.
R_n = remainder (what? Wait P248)

The larger n is, the more terms in P_n, & maybe the smaller R_n is.

R_n tells us P_n is an approximation.

R_n = measure of error.

eg $n=1$

$$\phi(x) = \overbrace{\left[\phi(x_0) + \phi'(x_0)(x-x_0)\right]}^{P_1} + R_1$$

$$= P_1 + R_1$$

P_1 is a linear approximation to $\phi(x)$. = has 2 terms

If $n=2$, we get a quadratic approx. = has 3 terms

Why do we like this?

- poly fns are easy to work with.
- can be good approximations - if add enough terms (maybe)

eg $\phi(x) = \dfrac{1}{1+x}$

Expand $\phi(x)$ around $x_0 = 1$, with $n = 4$

$$\phi'(x) = -(1+x)^{-2} \qquad \phi'(1) = -\frac{1}{4}$$

$$\phi''(x) = 2(1+x)^{-3} \qquad \phi''(1) = \frac{1}{4}$$

$$\phi'''(x) = -6(1+x)^{-4} \qquad \phi'''(1) = -\frac{3}{8}$$

$$\phi^{(4)}(x) = 24(1+x)^{-5} \qquad \phi''''(1) = \frac{3}{4}$$

Also $\phi(1) = \dfrac{1}{2}$

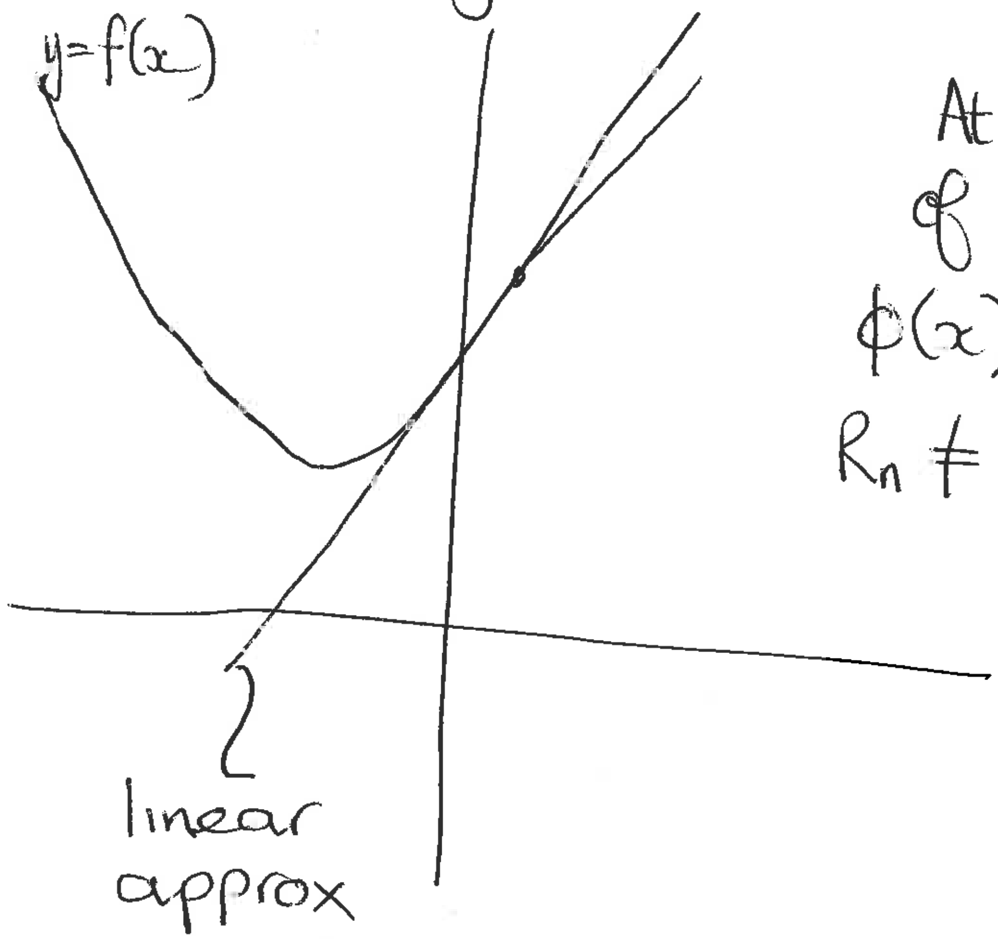
$$\text{Taylor Series} = \sum_{i=1}^{n} \frac{f^{(i)}(x_0)}{i!} \cdot (x - x_0)^i$$

$$= \phi(1) + \phi'(1)(x-1) + \frac{\phi''(1)(x-1)^2}{2} + \frac{\phi^{(3)}(1)(x-1)^3}{3!} + \frac{\phi^{(4)}(1)(x-1)^4}{4!} + R_4$$

$$\phi(x) = \frac{31}{32} - \frac{13}{16}x + \frac{1}{2}x^2 - \frac{3}{16}x^3 + \frac{1}{32}x^4 + R_4$$

eg 4 P246 = HW
& workshop

See diagram P247



At the pt of tangency $\phi(x) = P_n$ ($R_n = 0$)
$R_n \neq 0$ every where else

(*) If you are interested in value around $x = x_0$, then expand around that value of x, as P_n will be most accurate at that pt.

For now, we leave out
pages 248 – 254
We may return.
Watch this space