CHAPTER 08

ECO 4112 F

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Ch 8

$y = f(x_1, x_2, \ldots x_n)$

$\frac{\partial y}{\partial x_j}$ = derivative of y w.r.t x_j, while holding all the other x's constant.

- What if we cant? ie if x_2 changes when x_3 changes?

If we dont have explicit functional forms, we will also not able to solve for our y's in terms of x's – we will have interdependence!

eg: $Y = C + I_0 + G_0$
$C = C(Y, T_0)$ T_0 (exog tax)

$Y = C(Y, T_0) + I_0 + G_0$

Solving for Y* as an explicit fn is not possible.

Assume Y^* exists, under certain conditions:

$$Y^* = Y^*(I_0, G_0, T_0)$$

& Y^* is differentiable.

In some neighbourhood around Y^*, the identity holds:

$$Y^* = C(Y^*, T_0) + I_0 + G_0$$

= Eqm identity.

What is $\frac{\partial Y^*}{\partial T_0}$?

Y^* is a fn of T_0,

$C(Y^*, T_0)$ has 2 interdependen arguments.

T_0 affects C directly &

T_0 affects C indirectly through Y

We need total differentiation.

Differentials

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} \neq \frac{dy}{dx}$$

We can say

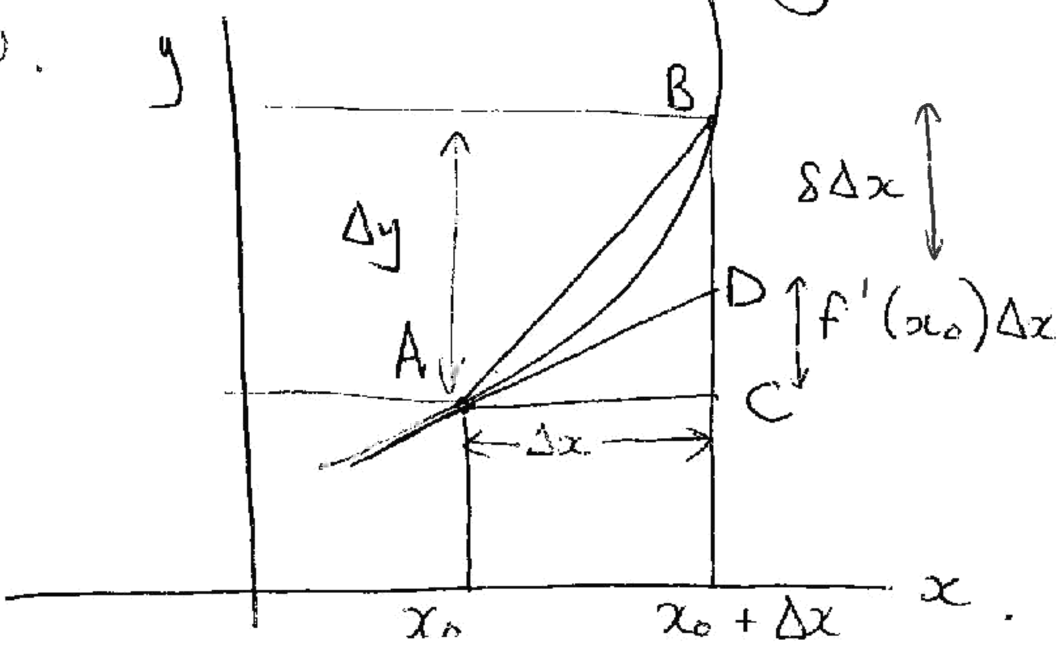
$$\frac{\Delta y}{\Delta x} - \frac{dy}{dx} = \delta \quad \text{where } \delta \to 0 \text{ as } \Delta x \to 0$$

$\times$ by Δx

$$\Delta y = \frac{dy}{dx} \cdot \Delta x + \delta \Delta x$$

$$\Delta y = f'(x) \cdot \Delta x + \delta \Delta x \qquad (8.3)$$

$f'(x)\Delta x$ approximates Δy, as $\Delta x \to 0$.



We have $y = f(x)$

Slope of $AD = f'(x)$

$CB = \Delta y$

$CB = CD + BD$

$$\frac{CB}{AC} = \frac{\Delta y}{\Delta x}$$

$CD = f'(x_0)\Delta x$ (why? $\frac{DC}{AC} = \underset{AD}{\text{slope}}$)

$\therefore DC = AC \cdot f'(x_0)$

$= \Delta x \cdot f'(x_0)$

We know from 8.3

$\Delta y = f'(x)\Delta x + \delta \Delta x$

$\therefore \Delta y = DC + \delta \Delta x$

$\therefore DB = \delta \Delta x$

As $\Delta x \to 0$, B slides towards A

$\therefore \Delta y \to f'(x)\Delta x$

$\therefore f'(x)$ becomes a better approx to $\frac{\Delta y}{\Delta x}$

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relabel AC & CD by dx & dy (CD is approximation to CB as $\Delta x \to 0$)

$$\frac{dy}{dx} = \text{slope tangent } AD = f'(x)$$

$$dy = f'(x)\,dx$$

- dy, dx = differentials,
- If we know dx, multiply by $f'(x)$ to get dy
- We have just shown it is ok to separate dy, dx

  Previously, only talked about $\frac{dy}{dx}$ as 1 entity.
- dy is dependent, dx independent.
- $dy = \text{fn}(x, dx) = f'(x)\,dx$

  this is a fn of x
- If $dx = 0$, $dy = 0$.
- $f'(x)$ converts change dx into a change dy.

Differentials & Elasticity

DD fn : $Q = f(P)$

elasticity $\varepsilon_d = \frac{\Delta Q}{Q} \Big/ \frac{\Delta P}{P}$

point elasticity $\varepsilon_d = \frac{dQ}{Q} \Big/ \frac{dP}{P}$

ie as $\Delta P \to 0$

$$= \frac{dQ}{dP} \Big/ \frac{Q}{P}$$

$$= \frac{\text{marginal fn}}{\text{average fn}}$$

Remember

$\varepsilon = 1$ unit elasticity

$\varepsilon < 1$ inelastic

$\varepsilon > 1$ elastic

P182 eg 1

P182 eg 2 : S fn: $Q = P^2 + 7P$
Is supply elastic at $P = 2$?

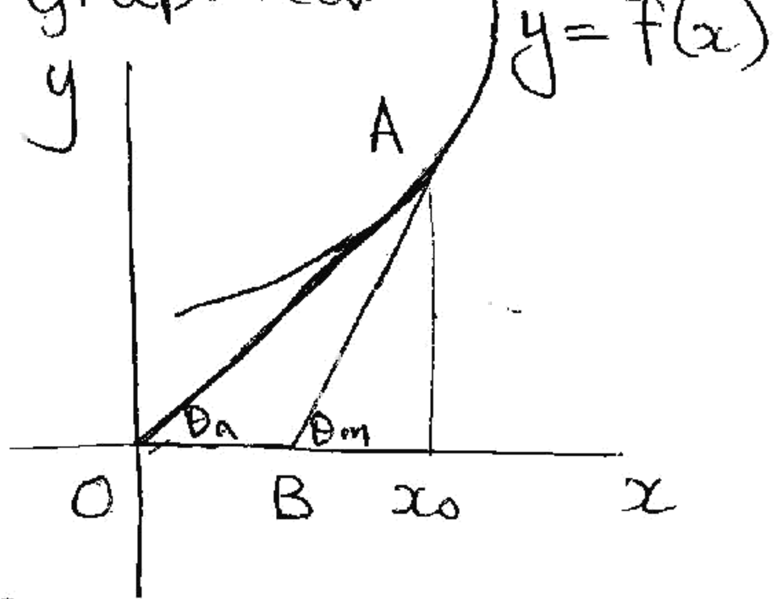
$\frac{dQ}{dP} = 2P + 7$ $\quad Q/P = P + 7$

$\varepsilon_s = \frac{2P+7}{P+7} = \frac{11}{9}$ when $P = 2$

elastic

P183 (7)

Using $\varepsilon = \dfrac{\text{marginal}}{\text{avg}}$ gives a good graphical way to see pt elasticity



AB is tangent to $y = f(x)$

$\therefore$ Slope AB = marginal fn

Slope OA = avg fn. Why?

At A, $y = x_0 A$ & $x = Ox_0$ (labelled distances)

slope of OA (ray from origin) $= \dfrac{x_0 A}{O x_0} = \dfrac{y}{x}$

Average fn $= \dfrac{y}{x}$

$\therefore$ elasticity at pt A:

if AB (marg) is steeper than OA (avg)

then elastic at A,

if AB flatter, inelastic.

Above: it is elastic

See other graphs P183 for comparison.

Also, if $\theta_m < \theta_a$

$\Rightarrow$ marg < avg

$\Rightarrow$ inelastic,

if $\theta_m > \theta_a$, elastic.

<u>Unit Elasticity?</u>

- At a pt where the tangent to f(x) passes through the origin & lies on the ray, = unit elasticity

  $\theta_m = \theta_a$.

- This assumes y is on vertical axis, ie for supply, we MUST have Q on the vertical axis.

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Revise 8.3 Rules of Differentials

Total Differentials

eg $S = S(Y, i)$

- Assume S is continuous & has continuous partial derivatives.
- for any change in Y, dy,

$dS = \frac{\partial S}{\partial Y} \cdot dY$ Similarly for di.

$$dS = \frac{\partial S}{\partial Y} \cdot dY + \frac{\partial S}{\partial i} \cdot di$$

If i doesnt change as Y changes, $di = 0$.

dS = total differential.

EG

$U = U(x_1, \ldots x_n)$

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \ldots + \frac{\partial U}{\partial x_n} dx_n$$

partial derivative

differential

$$= \sum_{i=1}^{n} U_i \, dx_i$$

$$U_i = \frac{\partial U}{\partial x_i}$$

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$U_i\, dx_i$ = marginal utility of good i × change in i consumed

dU = change in U from all possible sources of change.

$S = S(Y, i)$ now has 2 elasticities

$U = U(x_i, i = 1, 2, \ldots n)$ has n "

$$\varepsilon_{Ux_i} = \frac{\partial U}{\partial x_i} \cdot \frac{x_i}{U} \qquad i = 1, 2, \ldots n.$$

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eg $U(x_1, x_2) = x_1^a x_2^b$

$$dU = \underbrace{\left(a x_1^{a-1} x_2^b\right)}_{\frac{\partial U}{\partial x_1}} dx_1 + \underbrace{\left(b x_1^a x_2^{b-1}\right)}_{\frac{\partial U}{\partial x_2}} dx_2$$

$$\varepsilon_{Ux_2} = \frac{\partial U}{\partial x_2} \Big/ \frac{U}{x_2}$$

$$= b x_1^a x_2^{b-1} \Big/ \frac{x_1^a x_2^b}{x_2} = b$$

Rules of Differentials

k = constant , u, v = fns(x_1, x_2)

$$\left.\begin{aligned} &(1) \quad dk = 0 \\ &(2) \quad d(cu^n) = cnu^{n-1}\,du \\ &(3) \quad d(u \pm v) = du \pm dv \\ &(4) \quad d(uv) = v\,du + u\,dv \\ &(5) \quad d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \end{aligned}\right\} \text{no proofs given}$$

Introduce $w = \text{fn}(x_1, x_2)$

(6) $d(u \pm v \pm w) = du \pm dv \pm dw$

(7) $d(uvw) = vw\,du + uv\,dw + uw\,dv$

ex $\quad y = \dfrac{x_1 + x_2}{2x_1^2}$

$$dy = \frac{\partial y}{\partial x_1} \cdot dx_1 + \frac{\partial y}{\partial x_2} \cdot dx_2$$

$$= \left(\frac{-(x_1 + 2x_2)}{2x_1^3}\right) dx_1 + \left(\frac{1}{2x_1^2}\right) dx_2$$

(*) Can prove (6) & (7) using (1) - (5)

Total Derivatives

↳ do not require x_1 to stay constant when x_2 changes, if $y = f(x_1, x_2)$

eg $y = f(x, w)$ & $x = g(w)$

$\therefore$ $y = f(g(w), w)$

w affects y directly, & indirectly through x

$$\begin{tikzcd} \boxed{w} \arrow[r, "g"] \arrow[rr, bend right, "f"'] & \boxed{x} \arrow[r, "f"] & \boxed{y} \end{tikzcd}$$

$$dy = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial w}\, dw$$

$$\frac{dy}{dw} = \frac{\partial f}{\partial x}\frac{dx}{dw} + \frac{\partial f}{\partial w}$$

$$= \underbrace{f_x\, g'}_{\text{indirect}} + \underbrace{f_w}_{\text{direct effect}}$$

eg $y = f(x, w) = 3x - w^2$

$x = g(w) = 2w^2 + w + 4$

$$\frac{dy}{dw} = \frac{\partial y}{\partial x} \cdot \frac{dx}{dw} + \frac{\partial y}{\partial w}$$

$$= 3 \cdot (4w + 1) + (-2w)$$

$$= 12w + 3 - 2w = 10w + 3$$

eg $y = f(x_1, x_2, w)$ $\quad x_1 = g(w)$

$\quad x_2 = h(w)$

$$dy = f_1\, dx_1 + f_2\, dx_2 + f_w\, dw$$

$$\frac{dy}{dw} = \underbrace{f_1 \cdot \frac{dx_1}{dw}}_{\text{indirect}} + \underbrace{f_2 \frac{dx_2}{dw}}_{\text{indirect}} + \underbrace{f_w}_{\text{direct}}$$

eg $y = f(x_1, x_2, u, v)$

$x_1 = g(u, v)$

$x_2 = h(u, v)$

$$dy = \frac{\partial y}{\partial x_1}.dx_1 + \frac{\partial y}{\partial x_2}.dx_2 + \frac{\partial y}{\partial u}.du + \frac{\partial y}{\partial v}.dv$$

$$\therefore \frac{dy}{du} = \frac{\partial y}{\partial x_1}\frac{dx_1}{du} + \frac{\partial y}{\partial x_2}.\frac{dx_2}{du} + \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}.\frac{dv}{du}.$$

We hold v constant to find total derivative of y w.r.t u

$$\therefore \frac{dv}{du} = 0$$

Which derivatives are partials?

- $\frac{dy}{du}$ = partial total derivative denoted $\delta y/\delta u$
- $\frac{dx_1}{du}, \frac{dx_2}{du}$ are partials

$$\therefore \frac{\delta y}{\delta u} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial u} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial u} + \frac{\partial y}{\partial u}.$$

Similarly for partial total derivative $\frac{dy}{dv}$

Notes P193

(1) total derivatives are expressions of the chain rule (fns of fns of fns)

(2) Can extend to more than 3 fns

(3) total derivatives measure rates of change w.r.t final variables (exog vars)

Implicit functions

$y = f(x)$ = explicit function

$y - f(x) = 0$ implicit fn, which implies explicit $y = f(x)$

Generally $F(y, x) = 0$

f = explicit fn - 2 arguments

F = implicit fn - 1 argument

OR $F(y, x_1, x_2, \ldots x_n) = 0$

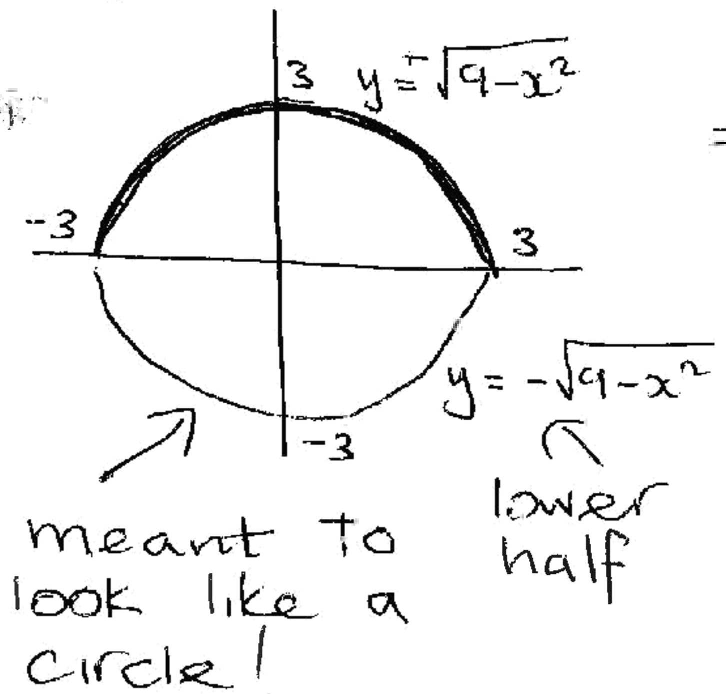
with implicit fn $y = f(x_1, x_2, \ldots x_n)$

- We can't always guarentee $y = f(x)$ is defined.

eg $x^2 + y^2 = 0$ is satisfied only at $(0,0)$, $\therefore$ there is no fn $y = f(x)$

eg $F(y,x) = x^2 + y^2 - 9 = 0$

$x^2 + y^2 = 9$
$=$ circle



This is a <u>relation</u>, not a fn – no unique value of y for each x value.

(FAILS the vertical line test)

So...

Are there known conditions s.t

$$F(y, x_1, \ldots, x_m) = 0$$

defines

$$y = f(x_1, \ldots x_m)?$$

Around some pt in the domain of x.

Implicit Function Theorem

Given F as above, if

(a) F has continuous partial derivatives $F_y, F_1, \ldots, F_m$ & if

(b) at a point $(y_0, x_{10}, \ldots, x_{m0})$ satisfying $F(y, x_1, \ldots x_m) = 0$,

$F_y \neq 0$ then...

there exists a neighbourhood N (m dimensional) in which

$y = f(x_1, \ldots x_m)$ is defined.

- This fn satisfies $y_0 = f(x_{10}, \ldots x_{m0})$

Also mtuples in neighbourhood N

$$F(y, x_1, \ldots x_m) \equiv 0$$

Also f is continuous, & has continuous partial derivatives.

eg $F(y, x) = x^2 + y^2 - 9 = 0$

Check

(a) $F_y = 2y$ are continuous
$F_x = 2x$

(b) $F_y \neq 0$ if $y \neq 0$.

Around any other pt excluding but btwn $(-3, 0)$ & $(3, 0)$ there is a neighbourhood where $y = f(x)$ is defined.

NOTES

L conditions are sufficient, but not necessary.
eg if $F_y = 0$, may still be a fn f defined at that pt.

L even if f exists, we dont know its specific form, or size of N.

L still helpful if dont have explicit fn f, & guarentees existence of $f_1, f_2, \ldots f_m$.

- HOLD UP! Proof of IF? No.
- Given $F(y, x_1, \ldots x_m) = 0$, & we know (using IF theorem) that $y = f(x_1, \ldots x_m)$ exists, but we can't solve for it — we can still find $f_1, f_2, \ldots f_m$. How?

<u>FACTS</u>

① If $A \equiv B$ then $dA \equiv dB$

② differentiate an expression with $y, x_1, \ldots x_m$, we get an expr with $dy, dx_1, \ldots dx_m$

③ We can substitute for dy.

ok if can't solve for y

$$F(y, x_1, \ldots x_m) \equiv 0$$

$$F_y dy + F_1 dx_1 + F_2 dx_2 + \ldots + F_m dx_m = 0$$

$$y = f(x_1, \ldots x_m)$$

$$dy = f_1 dx_1 + \ldots + f_m dx_m.$$

Substitute this in for dy above.

$$\therefore F_y (f_1 dx_1 + f_2 dx_2 + \ldots f_m dx_m) + F_1 dx_1 + F_2 dx_2 + \ldots + F_m dx_m = 0$$

$$\therefore (F_y f_1 + F_1) dx_1 + (F_y f_2 + F_2) dx_2 + \ldots + (F_y f_m + F_m) dx_m = 0$$

For this to hold, we need that for each variable i,

$$F_y f_i + F_i = 0 \qquad \text{Why?}$$

$$\therefore f_i = -\frac{F_i}{F_y}$$

$$\therefore \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}$$

$\therefore$ for $F(y, x) = 0$

we get $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$

Now we see why we need $F_y \neq 0$.

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eg

Find $\frac{\partial y}{\partial x}$ for an IF defined

by $F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = ($

We know $\frac{\partial y}{\partial x} = \frac{-F_x}{F_y}$

$$\therefore \frac{\partial y}{\partial x} = \frac{-(2y^3x + yw)}{3y^2x^2 + xw}$$

Wait Did we check condition of IF hold?

We need (a) F_y, F_x, F_w to be continuous & to exist

(b) At a pt satisfying $F(y, x, w) = 0$, $F_y \neq 0$

then $y = f(x, w)$ is defined around that point & it's okay to use the rule to find $\frac{\partial y}{\partial x}$

(22)

$\therefore$ for $y^3x^2 + w^3 + yxw - 3 = 0$

(a) We have $F_y = 3y^2x^2 + xw$

$F_x = 2y^3x^2 + yw$

$F_w = 3w^2 + yx$

$\therefore$ continuous & exist

(b) find a pt satisfying $F = 0$

eg $(1,1,1)$

$(1)^3(1)^2 + (1)^3 + (1)(1)(1) - 3 = 0$

$0 = 0$

$\therefore$ At $(1,1,1)$ $F_y = 3(1)^2(1)^2 + (1)(1)$

$= 4 \neq 0$

$\therefore$ $y = f(x,w)$ exists around $(1,1,1)$ & we can talk about $\dfrac{\partial y}{\partial x}$.

Did we solve for $y = f(x,w)$?

No.

eg $F(Q, K, L)$ implicitly defined production fn $Q = f(K, L)$

- We want to know MPP_K, MPP_L (why not just MP_K, MP_L)?
- Assume F_Q, F_L, F_K exist & are continuous, & for a point which satisfies $F(Q, K, L) = 0$, $F_Q \neq 0$.

Then $Q = f(K, L)$ exists, &

$$\frac{\partial Q}{\partial L} = MPP_L = -\frac{F_L}{F_Q}$$

& $$\frac{\partial Q}{\partial K} = MPP_K = -\frac{F_K}{F_Q}$$

Also $$\frac{\partial K}{\partial L} = -\frac{F_L}{F_K}$$ what is this expr?

How we can change K & L, while keeping Q constant ie slope of the isoquant $\frac{\partial K}{\partial L} < 0$

But $\left|\frac{\partial K}{\partial L}\right| = MRTS_{KL}$

Simultaneous EQN Case

aka life is never easy.

Given n functions F^1 to F^n:

$$F^1(y_1, y_2, \ldots y_n, x_1, \ldots x_m) = 0$$
$$F^2(y_1, y_2, \ldots y_n; x_1, \ldots x_m) = 0$$
$$\vdots$$
$$F^n(y_1, y_2, \ldots y_n; x_1, \ldots x_m) = 0$$

Under what conditions do these define:

$$y_1 = f^1(x_1, \ldots x_m)$$
$$y_2 = f^2(x_1, \ldots x_m)$$
$$\vdots$$
$$y_n = f^n(x_1, \ldots x_m)$$

IF Theorem

Given the set of F^i above, if (a) $F^1 \ldots F^n$ all have continuous partial derivatives w.r.t all y & x, & if (b) at a point satisfying $F^i = 0$, $|J| \neq 0$ then $y_i = f^i(x_1, \ldots x_m)$ is defined

(25)

The point is $(y_{10}, y_{20}, \ldots y_{m0}; x_{10}, \ldots x_{m0})$
The $|J|$ is $\left| \dfrac{\partial(F^1, \ldots F^n)}{\partial(y_1, \ldots y_n)} \right|$

Note w.r.t y's only.

$$\therefore |J| = \begin{vmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & & & \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{vmatrix} \quad n \times n$$

We need $|J| \neq 0$ for the pt satisfying the $F^i = 0$ above, & if so, there exists an m-dimensio[n] neighbourhood N of $(x_{10}, x_{20}, \ldots x_{m0})$ in which $y_1, \ldots, y_n$ are fns of $x_1, \ldots x_m$ as above, & these fns satisfy

$$y_{10} = f^1(x_{10}, \ldots x_{m0})$$

$$y_{n0} = f^n(x_{10}, \ldots, x_{m0})$$

$\Rightarrow$ the F^i are identities
$\Rightarrow$ f^i are continuous, have cont partial deriv

- We dont have to solve for the y_i to find $\frac{\partial y_i}{\partial x_j}$ for any $\begin{matrix} i=1\ldots n \\ j=1\ldots m \end{matrix}$
- We can take the total differential of the $F^i = 0$
  $dF^j = 0$ for $j = 1, 2, \ldots, n$.

We get a set of equations with $dy_1, \ldots dy_n ; dx_1, \ldots dx_n$.

Specifically: for $F^i(y_1, \ldots y_n ; x_1, \ldots x_m) = 0$

- $$\frac{\partial F^1}{\partial y_1} dy_1 + \frac{\partial F^1}{\partial y_2} dy_2 + \ldots + \frac{\partial F^1}{\partial y_n} dy_n = -\left(\frac{\partial F^1}{\partial x_1} dx_1 + \frac{\partial F^1}{\partial x_2} dx_2 + \ldots \frac{\partial F^1}{\partial x_m} dx_m \right)$$
- $$\frac{\partial F^2}{\partial y_1} dy_1 + \frac{\partial F^2}{\partial y_2} dy_2 + \ldots + \frac{\partial F^2}{\partial y_n} dy_n = -\left(\frac{\partial F^2}{\partial x_1} dx_1 + \ldots + \frac{\partial F^2}{\partial x_m} dx_m \right)$$

$\vdots$

$$\frac{\partial F^n}{\partial y_1} dy_1 + \frac{\partial F^n}{\partial y_2} dy_2 + \ldots + \frac{\partial F^n}{\partial y_n} dy_n = -\left(\frac{\partial F^n}{\partial x_1} dx_1 + \ldots + \frac{\partial F^n}{\partial x_n} dx_n \right)$$

We can write the dy_i from $y_i = f(x_1, \ldots)$

$$dy_1 = \frac{\partial y_1}{\partial x_1} dx_1 + \frac{\partial y_1}{\partial x_2} dx_2 + \cdots + \frac{\partial y_1}{\partial x_m} dx_m$$

$$dy_2 = \frac{\partial y_2}{\partial x_1} dx_1 + \frac{\partial y_2}{\partial x_2} dx_2 + \cdots + \frac{\partial y_2}{\partial x_m} dx_m$$

$$\vdots$$

$$dy_n = \frac{\partial y_n}{\partial x_1} dx_1 + \frac{\partial y_n}{\partial x_2} dx_2 + \cdots + \frac{\partial y_n}{\partial x_m} dx_m$$

We substitute these into the expressions above for $dF^i = 0$.

Eg: for the 1st eqn: Set $dx_1 \neq 0$
other $dx_i = 0$

$$\frac{\partial F^1}{\partial y_1}\left(\frac{\partial y_1}{\partial x_1} dx_1 + \frac{\partial y_1}{\partial x_2} dx_2 + \cdots + \frac{\partial y_1}{\partial x_m} dx_m\right)$$

$$+ \frac{\partial F^1}{\partial y_2} dy_2 + \cdots \frac{\partial F^1}{\partial y_n} = -\left(\frac{\partial F^1}{\partial x_1} . dx_1\right)$$

Collect terms:

$$\frac{\partial F^1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1} + \frac{\partial F^1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1} + \cdots + \frac{\partial F^1}{\partial y_n}\cdot\frac{d y_n}{\partial x_1} = -\frac{\partial F^1}{\partial x_1}$$

$$\frac{\partial F^2}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial F^2}{\partial y_2}\frac{\partial y_2}{\partial x_1} + \cdots + \frac{\partial F^2}{\partial g_n}\cdot\frac{\partial y_n}{\partial x_1} = -\frac{\partial F^2}{\partial x_1}$$

$$\vdots$$

$$\frac{\partial F^n}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1} + \frac{\partial F^n}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1} + \cdots + \frac{\partial F^n}{\partial y_n}\cdot\frac{\partial y_n}{\partial x_1} = -\frac{\partial F^n}{\partial x_1}$$

WAIT

Why can we set $dx_1 \neq 0$, $dx_i = 0$ if $i \neq 1$?

We want $\frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_1} \cdots \frac{\partial y_n}{\partial x_1}$

ie derivatives of y's w.r.t x_1 holding other x_i's constant

ie dx_i's $= 0$ Why?

- If we dont set other dx_i's $= 0$ We have a huge mess.
- Looking at our previous system

It has $\frac{\partial y_i}{\partial x_1}$ — these we want to solve for.

It has $\frac{\partial F^i}{\partial y_j}$'s. At pt satisfying the F^is $= 0$, these derivatives are constants.

All in all, we get a linear system

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & & & \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_1} \\ \frac{\partial y_2}{\partial x_1} \\ \vdots \\ \frac{\partial y_n}{\partial x_1} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial x_1} \\ -\frac{\partial F^2}{\partial x_1} \\ \vdots \\ -\frac{\partial F^n}{\partial x_1} \end{bmatrix}$$

$$n \times n \qquad\qquad n \times 1 \qquad\qquad n \times 1$$

The determinant of this mx is $|J|$, & it is assumed $\neq 0$ if IF conditions are met.

This is a nonhomogenous system (if RHS $-\frac{\partial F^i}{\partial x_1}$ were all $= 0$, with $|J| \neq 0$, only solution is trivial:

$\frac{\partial y_i}{\partial x_1} = 0$ = which is not very interesting!

∴ there should be a unique nontrivial solution.

<u>Cramer's Rule</u> $\frac{\partial y_j}{\partial x_1} = \frac{|J_j|}{|J|}$ $j = 1, 2, \ldots, n$

- We can also get the other partial derivatives w.r.t $x_2, \ldots x_m$ using same procedure.
- The matrix J will not change however, only vectors

eg: Want $\frac{\partial y_i}{\partial x_3}$

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \vdots & & & \\ \vdots & & & \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_3} \\ \vdots \\ \frac{\partial y_n}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial x_3} \\ -\frac{\partial F^2}{\partial x_3} \\ \vdots \\ -\frac{\partial F^n}{\partial x_3} \end{bmatrix}$$

- Remember: $|J|$ has derivatives of F^i only w.r.t the $\underline{y_i}$
- Constraining $|J| \neq 0 \Rightarrow$ Cramer's Rule will yield <u>proper</u> answers

<u>EG</u>

$$xy - w = 0 \qquad F^1(x, y, w, z) = 0$$
$$y - w^3 - 3z = 0 \qquad F^2(x, y, w\,; z) = 0$$
$$w^3 + z^3 - 2zw = 0 \qquad F^3(x, y, w\,; z) = 0$$

These equations are satisfied at $(\underset{x}{\tfrac{1}{4}}, \underset{y}{4}, \underset{w}{1}, \underset{z}{1})$ = point P

Do the F^i fns have continuous derivatives? Yes

If $|J| \neq 0$ at point P, then according to IFT, we can find $\dfrac{\partial x}{\partial z}$.

✱ <u>WAIT</u>

- how to find P? Trial & Error
- Is P unique? Probably not

- Why $\dfrac{\partial x}{\partial z}$?

z = exog, we will find $\dfrac{\partial y}{\partial z}$ & $\dfrac{\partial w}{\partial z}$ too

(32)

We take differential (total) of each eqn:

$$(y)dx + (x)dy + (-1)dw + (0)d\!:\ = 0$$

$$(0)dx + (1)dy + (-3w^2)dw + (-3)\text{c} = 0$$

$$(0)dx + (0)dy + (3w^2 - 2z)dw + (3z^2 - 2w)d\text{.} = 0$$

Move dz terms to RHS

$$\begin{bmatrix} y & x & -1 \\ 0 & 1 & -3w^2 \\ 0 & 0 & 3w^2-2z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dw \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ (2w - 3z^2) \end{bmatrix} dz$$

Check, at pt P, $|J| \neq 0$

$|J| = y(3w^2 - 2z) = 3w^2y - 2zy = 4$ at pt P.

∴ IFT holds, & y: are defined around P

$$\begin{bmatrix} & & \\ & J & \\ & & \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial z} \\ \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2w - 3z^2 \end{bmatrix}$$

We divided through by dz, we get partial derivatives of y_i

What is $\frac{\partial x}{\partial z}$? $= \frac{|J_1|}{|J|}$

$$= \frac{\begin{vmatrix} 0 & x & -1 \\ 3 & 1 & -3w^2 \\ 2w-3z^2 & 0 & 3w^2-2z \end{vmatrix}}{|J|} = -\frac{1}{4} \text{ at pt } P.$$

✱ do you know how to get $\frac{\partial w}{\partial z}$?

eg NI model

$$Y - C - I_0 - G_0 = 0$$
$$C - \alpha - \beta(Y - T) = 0$$
$$T - \gamma - \delta Y = 0$$

Y, C, T = endog

exog = $I_0, G_0, \alpha, \beta, \gamma, \delta$

each LHS is : $F^i(Y, C, T, I_0, G_0, \alpha, \beta, \gamma, \delta) = 0$

To find $\frac{\partial Y, C \text{ or } T}{\partial I_0, G_0 \text{ or other exog}}$ must we find total differentials each time? No.

We know:

$$\underbrace{\left[\begin{array}{c} J \\ = \dfrac{\partial F^i}{\partial y_j} \quad {\scriptstyle i=1,\dots n \atop j=1,\dots n} \\ \text{derivative of } F^i \text{ w.r.t endog vars} \end{array}\right]}_{} \underbrace{\begin{bmatrix} \dfrac{\partial y_1}{\partial x_j} \\ \dfrac{\partial y_2}{\partial x_j} \\ \vdots \\ \dfrac{\partial y_n}{\partial x_j} \end{bmatrix}}_{\substack{= \text{derivatives of } y_i\text{'s w.r.t} \\ \text{exog var of interest}}} = \underbrace{\begin{bmatrix} -\dfrac{\partial F^1}{\partial x_j} \\ -\dfrac{\partial F^2}{\partial x_j} \\ \vdots \\ -\dfrac{\partial F^n}{\partial x_j} \end{bmatrix}}_{\substack{= -\text{derivative of } F^i \\ \text{w.r.t exog var of interest}}}$$

You should know why, but dont have to prove it every time. Assuming IFT conditions met, can start from here.

Back to eg

F^1, F^2, F^3 have continuous partial derivatives, w.r.t all endog & exog (usually show this) & check $|J| \neq 0$

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial Y} & \frac{\partial F^1}{\partial C} & \frac{\partial F^1}{\partial T} \\ \frac{\partial F^2}{\partial Y} & \frac{\partial F^2}{\partial C} & \frac{\partial F^2}{\partial T} \\ \frac{\partial F^3}{\partial Y} & \frac{\partial F^3}{\partial C} & \frac{\partial F^3}{\partial T} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{vmatrix}$$

$$= 1(1) + 1(-\beta + \beta\delta)$$

$$= 1 - \beta + \beta\delta > 0 \qquad (1-\beta > 0)$$

$\therefore$ We can take Y, C, T to be implicit fns of exog vars at/around a pt which satisfies the original set of eqns $F^i = 0$

$$\therefore \quad Y^* = f^1(I_0, G_0, \alpha, \beta, \gamma, \delta)$$
$$C^* = f^2(\;"\;\;"\;\;"\;\;"\;\;"\;\;"\;)$$
$$T^* = f^3(\;"\;\;"\;\;"\;\;"\;\;"\;)$$

We can now $\frac{\partial Y}{\partial I_0}$ or $\frac{\partial Y}{\partial G_0}$.

$$\begin{bmatrix} & & \\ & J & \\ & & \end{bmatrix} \begin{bmatrix} \frac{\partial Y}{\partial I_0} \\ \frac{\partial C}{\partial I_0} \\ \frac{\partial T}{\partial I_0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

eg $\frac{\partial T}{\partial I_0} = \frac{|J_3|}{|J|} = \frac{\begin{vmatrix} 1 & -1 & 1 \\ -B & 1 & 0 \\ -\delta & 0 & 0 \end{vmatrix}}{|J|}$

$$\frac{\partial T}{\partial I_0} = \frac{1(\delta)}{1-B+B\delta}$$

Previously, we solved for Y, C, T & then found derivatives.

L here used IFT to go straight to derivatives.

<u>CS of General Fn Models</u>

<u>Wait!</u> Did we, will we prove IFT in simultanea EQN case?

Market Model

Q_D = fn (Price, exog Y_0 - income)
Q_S = fn (price alone)

Generally:

$$Q_D = Q_S$$
$$Q_D = D(P, Y_0) \qquad \frac{\partial D}{\partial P} < 0 \quad \frac{\partial D}{\partial Y_0} > 0$$
$$Q_S = S(P) \qquad \frac{dS}{dP} > 0$$

- Assume D & S have continuous partial derivatives

Y_0 = exog inc.

- Can rewrite

$$D(P, Y_0) - S(P) = 0$$

Can't solve for $P^* = f(Y_0)$, but we assume P^* exists. Can we?

IFT single equation case:

Need (a) F_P, F_{Y_0} exist & be continuous ✓
(b) Need $F_P \neq 0$ (P = endog var)

$$F_P = \underset{-}{\frac{\partial D}{\partial P}} - \underset{-(+)}{\frac{dS}{dP}} < 0 \quad \therefore \neq 0$$

$\therefore$ can use IFT rule $\frac{\partial P}{\partial Y_0} = \frac{-F_{Y_0}}{F_P}$

Because IFT holds, we know

① $P^* = P^*(Y_0)$ is defined around a point B satisfying $F(P, Y_0) = 0$

② $D(P^*, Y_0) - S(P^*) \equiv 0$ (identity around B)

∴ Rule $\frac{dP^*}{dY_0} = -\frac{\frac{\partial F}{\partial Y_0}}{\frac{\partial F}{\partial P}} = \frac{-\frac{\partial D}{\partial Y_0}}{\frac{\partial D}{\partial P^*} - \frac{dS}{dP^*}}$

$$= \frac{-(+)}{(-) \quad -(+)} = \frac{-}{-} = +ve$$

• NB : Comparative Static derivative $= \frac{dP^*}{dY_0}$

= partial derivatives of the implicit fn/s, evaluated at EQM state

eg for $Y, C, T = \text{fns}(I_0, G_0, \alpha, \beta, \gamma, \delta)$

$\frac{\partial Y^*}{\partial I_0}, \frac{\partial C^*}{\partial I_0}, \frac{\partial T^*}{\partial I_0}$ are comparative static derivatives.

Can we find $\frac{dQ^*}{dY_0}$?

We have $Q^* = S(P^*)$ in EQM
& $P^* = P^*(Y_0)$

$$\frac{dQ^*}{dY_0} = \underset{(+)}{\frac{dS}{dP^*}} \cdot \underset{(+)}{\frac{dP^*}{dY_0}} > 0$$

we just showed this.

- If we knew the functional forms & actual EQM values, we could find the values of the derivative above
- What do results mean? If raise Y_0, this shifts D up,
  ⇒ EQM price rises, as does quantity
- Here we've got a general formula.

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Can we study P^* & Q^* simultaneously?

From 8.3.2

$Q_D = Q_S$

$Q_D = D(Y_0, P)$

$Q_S = S(P)$

Let $Q_D = Q_S = Q$

$$F^1(P, Q, Y_0) = D(P, Y_0) - Q = 0$$

$$F^2(P, Q, Y_0) = S(P) \quad - Q = 0$$

What are conditions of IFT in simul. eqn case?

(a) Continuous partial derivatives

↳ assumed from derivatives of Q_D, Q_S fns given.

(b) Need $|J| = 0$ $\quad J =$ mx of partials w.r.t endog vars

$$\begin{vmatrix} \frac{\partial F^1}{\partial P} & \frac{\partial F^1}{\partial Q} \\ \frac{\partial F^2}{\partial P} & \frac{\partial F^2}{\partial Q} \end{vmatrix} = \begin{vmatrix} \frac{\partial D}{\partial P} & -1 \\ \frac{dS}{dP} & -1 \end{vmatrix} = \underset{(+)}{\frac{dS}{dP}} - \underset{(-)}{\frac{\partial D}{\partial P}} > 0$$

Therefore, if EQM pt (P^*, Q^*) exists, we can write

$P^* = P^*(Y_0)$, $Q^* = Q^*(Y_0)$

(our implicit functions)

& $F^1(P^*, Q^*, Y_0) \equiv 0$ identities
$F^2(P^*, Q^*, Y_0) \equiv 0$

$\therefore$ can take total differentials move terms with dY_0 to RHS, ÷ through by dY_0 & we will get this:

$$\begin{bmatrix} & & \\ & J & \\ & & \end{bmatrix} \begin{bmatrix} \frac{dP^*}{dY_0} \\ \\ \frac{dQ^*}{dY_0} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial Y_0} \\ \\ -\frac{\partial F^2}{\partial Y_0} \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & J & \\ & & \end{bmatrix} \underbrace{\begin{bmatrix} \\ \text{"} \\ \\ \end{bmatrix}}_{\text{CS derivatives}} = \begin{bmatrix} -\frac{\partial D}{\partial Y_0} \\ \\ 0 \end{bmatrix}$$

CS derivatives

∴ use Cramer's Rule to solve:

$$\frac{dP^*}{dY_0} = \frac{|J_1|}{|J|} = \frac{\begin{vmatrix} -\frac{\partial D}{\partial Y_0} & -1 \\ 0 & -1 \end{vmatrix}}{|J|} = \frac{\frac{\partial D}{\partial Y_0}}{|J|}$$

$$\frac{dQ^*}{dY_0} = \frac{|J_2|}{|J|} = \frac{\begin{vmatrix} \frac{\partial D}{\partial P^*} & -\frac{\partial D}{\partial Y_0} \\ \frac{dS}{dP^*} & 0 \end{vmatrix}}{|J|} = \frac{\frac{\partial D}{\partial Y_0} \cdot \frac{dS}{dP^*}}{|J|}$$

Are these the same as previous answers in single EQN case? Yes.

P209 We could also just take the total derivative w.r.t Y_0 of the identities

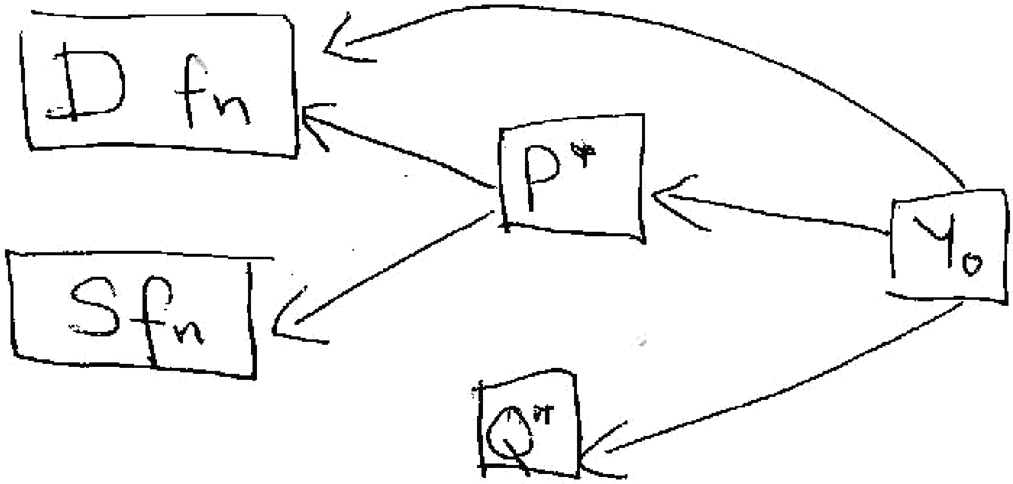
eg $D(P^*, Y_0) - S(P^*) \equiv 0$ $(P^* = P^*(Y_0))$

$$\frac{\partial D}{\partial P^*} \cdot \frac{dP^*}{dY_0} + \frac{\partial D}{\partial Y_0} - \frac{dS}{dP^*} \cdot \frac{dP^*}{dY_0} = 0$$

Why? Look back to P192

We got:

indirect effect of Y_0 on D + direct effect of Y_0 on D − indirect effect of Y_0 on S



- Is this a touch confusing?
- Can do same with simultaneous eqn case to get same answers as before.
- I want you to be able to name the effects as direct & indirect

National Income Model

ISLM model − want EQM in goods & money mkts

we want $Y^* = Y^*(G_0, M_0^s)$

$r^* = r^*(G_0, M_0^s)$

How do we know which are endog / exog variables?
Context / given.

$\boxed{\text{Example P211}}$ Closed model
↳ self study, workshop.

Open Economy

Given:

$$\boxed{\begin{aligned} Y &= C(Y^D) + I(r) + G_0 + X(E) - M(Y,E) \\ L(Y,r) &= M^s_0 \\ X(E) - M(Y,E) &+ K(r, r_w) = 0 \end{aligned}}$$

3 EQNS, 3 endog $= Y, r, E$

exog $= G_0, M^s_0, r_w$

Given / We know

- $X = X(E)$ has $X'(E) > 0$
  X = exports, are an increasing fn of the exchange rate E (domestic price of foreign currency)
- $M = M(Y,E)$ $\quad M_Y > 0, M_E < 0$
  M = imports

E = for eg, rand dollar exchang
ie 8 rand to the dollar

if E ↑, we import less.

- $K = K(r, r_w)$ K = net inflow of capital to a country

r = domestic int rate (endog)
r_w = world int rate (exog)

∴ $K_r > 0$ $K_{r_w} < 0$. Why?

- BOP

$$\boxed{BP = \underbrace{\text{current acc}}_{X(E) - M(Y,E)} + \underbrace{\text{capital acc}}_{K(r, r_w)}}$$

(net export) (purchase of bonds - foreign & domestic)

if E = flexible,
BP = 0
(ie SS dollars = DD dollars by SA)

Open Economy Eqm

As 3 eqns in box P44.
Do they make sense?

$$\boxed{\begin{aligned} AD &= AS \\ MD &= MS \\ BP &= 0 \end{aligned}}$$

V complicated, be careful.

Can we find $\frac{\partial Y^*, r^*, E^*}{\partial \text{exog}}$? Only if IFT holds?

1. <u>Assume</u> continuous partial derivatives exist? (given this previously with assns about $X'(E)$, m_Y, m_E, K_r, K_{rw}, $C'(Y^D)$, $I'(r)$ & we know sum of continuous fn = cont fn, ∴ $\frac{\partial F^{1,2,3}}{\partial \text{exog, endog}}$ exist & are continuous.

2. We need $|J| \neq 0$

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial Y} & \frac{\partial F^1}{\partial r} & \frac{\partial F^1}{\partial E} \\ \frac{\partial F^2}{\partial Y} & \frac{\partial F^2}{\partial r} & \frac{\partial F^2}{\partial E} \\ \frac{\partial F^3}{\partial Y} & \frac{\partial F^3}{\partial r} & \frac{\partial F^3}{\partial E} \end{vmatrix} = \rho$$

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<u>BTW</u>

We also know / are given:

$C'(Y^D)$ btwn 0 & 1, $I'(r) < 0$ why?

$$|J| = \begin{vmatrix} \left[1 - C'(Y^D)(1-T'(Y)) + m_Y\right] & -I' & m_E - X' \\ L_Y & L_r & 0 \\ -m_Y & K_r & X' - m_E \end{vmatrix}$$

$\boxed{\text{Apologies}}$, forgot to define

Y^D = disposable inc

$= Y - T$

& $T = T(Y)$

$\therefore$ $C'(Y^D)$

is actually meant to be:

$$\frac{d}{dY}\left[C(Y - T(Y))\right] = C'(Y^D)\left[1 - T'(Y)\right]$$

by chain rule

$\therefore$ see above

So is $|J| \neq 0$

$$|J| = (m_E - X')(L_Y K_r + L_r m_Y)$$
$$+ (X' - m_E)\big(L_r(1 - c'(1-T') + m_Y)$$
$$+ I' L_Y\big)$$

(we used the 3rd col to expand)

(Please don't faint).

Is this $|J| \neq 0$?

we need to tidy up to tell.

factorise & collect terms

$$= (m_E - X')\{L_Y(K_r - I')$$
$$+ L_r(c'(1-T') - 1)\}$$

$$= ((-) - (+))\{(+)((+) - (-)) + (-)(-)\}$$

$$= (-)\{(+) + (+)\} < 0.$$

$$\therefore |J| < 0$$

$\therefore$ IFT conditions are met

& we can write

$$Y^* = Y^*(q_0, m_0^s, r_w)$$
$$r^* = r^*(\;"\;\;"\;\;"\;)$$
$$E^* = E^*(\;"\;\;"\;\;"\;)$$

We can write

$$\begin{bmatrix} & & \\ & J & \\ & & \end{bmatrix} \begin{bmatrix} \frac{\partial Y^*}{\partial r_w} \\ \frac{\partial r^*}{\partial r_w} \\ \frac{\partial E^*}{\partial r_w} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial r_w} \\ -\frac{\partial F^2}{\partial r_w} \\ -\frac{\partial F^3}{\partial r_w} \end{bmatrix}$$

We let dq_0, $dm_0^s = 0$

(Why w.r.t r_w? Question asks for this.)

3rd vector of $Ax = d$ is $\begin{bmatrix} 0 \\ 0 \\ -K_{r_w} \end{bmatrix}$

Now use Cramer's Rule to solve

(I will spare you this)

So....

Is this insane? Yes

Is it necessary? Yes

Why?

- you will encounter general systems like these, must know how to solve for derivatives needed
- illustration of techniques (determinants, Cramer's Rule, derivatives)
- Refresher on ISLM & assumptions of relationships (you're expected to know this)

S1

Assumptions were:

L_Y

L_r

$X'(E)$

m_Y

m_E

K_r

K_{rw}

$T'(Y)$

$C'(Y^D)(1 - T'(Y))$

$I'(r)$

Know signs, (sizes if possible)

S2 P216

<u>In summary</u>, single or simul eQN case

Set up $F(y, x_i^\circ) = 0$

OR $F^1(y_i^\circ, x_i^\circ) = 0$

$F^2(\text{"} \quad \text{"}) = 0$

$F^n(\text{"} \quad \text{"}) = 0$

Check either $F_y, F_1, \ldots F_m$ exist & are continuous

OR all $F^1_{y_1} \ldots F^1_{y_n}, F^1_{x_1} \ldots F^1_{x_m}$

$F^n_{y_1}, \ldots F^n_{y_n}, F^n_{x_1} \ldots F^n_{x_m}$ exist & are continua

Then check $F_y \neq 0$

OR $|J| \neq 0$

where $|J| = \dfrac{\partial F^i}{\partial \text{ endog vars}}$

Then proceed to solve for

$$\frac{\partial \text{ eqm values of endog vars}}{\partial \text{ exog}}$$ (set d other exog = 0)