ECO 4112F

Chapter 10.6

10.6 Optimal Timing

Suppose a wine dealer must decide to either sell a case of wine at $t=0$ for \$K, or store it & sell it later for more.

V = value of wine known to be $= Ke^{\sqrt{t}}$

$= Ke^{t^{1/2}}$

If $t=0$, $V = Ke^{0^{1/2}} = Ke^0 = K$

Assume storage costs = 0, cost of wine = sunk cost, want to maximise π = sales revenue = K

Each V is related to different pts in time – we need to discount them to PV

Assume int rate r

$\therefore \quad A(t) = Ve^{-rt} = Ke^{t^{1/2}} \cdot e^{-rt} = Ke^{\sqrt{t}-rt}$

So to find t which maximises A Set $\frac{dA}{dt} = 0$

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We 1st log both sides

$$A = Ke^{\sqrt{t} - rt}$$

$$\log A = \log K + \log e^{\sqrt{t}-rt}$$

$$= \log K + (\sqrt{t} - rt)\underbrace{\log e}_{(=1)}$$

$$\log A = \log K + \sqrt{t} - rt$$

$$\frac{1}{A}\frac{dA}{dt} = \frac{1}{2}t^{-1/2} - r$$

$$\frac{dA}{dt} = A\left(\frac{1}{2}t^{-1/2} - r\right) = 0$$

We know $A \neq 0$ (why? $K > 0$, $e^{\sqrt{t}-rt} > 0$)

$$\frac{1}{2}t^{-1/2} = r$$

$$\therefore t^{-1/2} = 2r$$

$$t = (2r)^{-2}$$

$$t^* = \frac{1}{4r^2}$$

eg if $r = 0.10$

$\Rightarrow t^* = 25$

The higher r is, the shorter the optimum storage is.

(1b)

From $\frac{1}{2}t^{-1/2} = r$ (setting $\frac{dA}{dt} = 0$)

We can interpret as:

rate of growth of wine value V. Why?

└ this is $\frac{dV}{dt}/V = r_V$

So $\frac{dV}{dt} = \frac{d}{dt}\left(Ke^{t^{1/2}}\right) = K \, e^{t^{1/2}} \, \frac{1}{2}t^{-1/2}$

$$\boxed{\text{Why? } \frac{d}{dt}\left(e^{f(t)}\right) = f'(t)e^{f(t)} \quad \underline{\text{Chain Rule}}}$$

$\therefore \frac{dV}{dt} = \frac{1}{2}t^{-1/2} \cdot V$

$r_V = \frac{dV}{dt}/V = \frac{1}{2}t^{-1/2} = r$

$\therefore$ $r_V =$ rate of growth of the value V of the wine

at $t \to \infty$, $r_V \to 0$

$t \to 0$, $r_V \to \infty$

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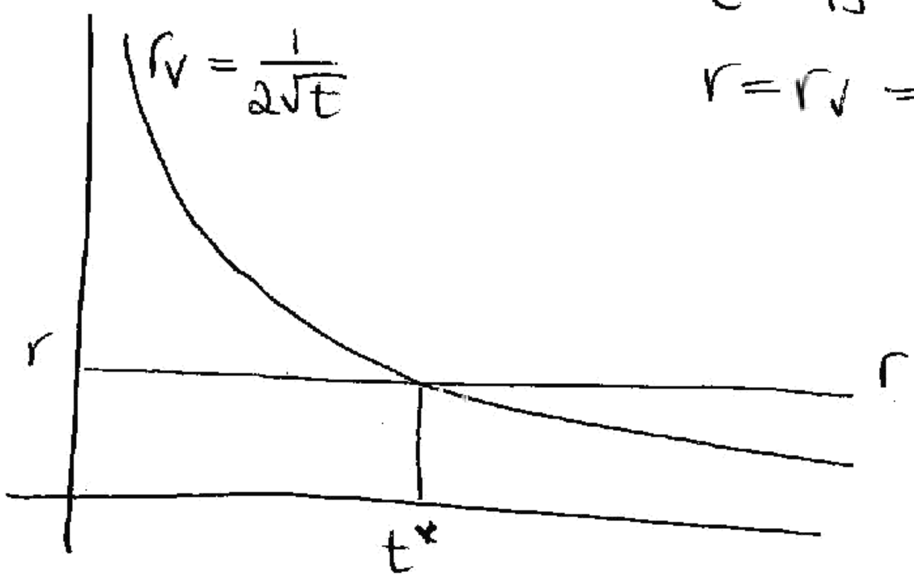
- r was rate of compounding interest of $ value of wine, if sold right away (ie at $t=0$)
- r reflects opportunity cost of keeping & storing the wine

$\boxed{10.4}$ P284



t^* is where
$r = r_V = \frac{1}{2\sqrt{t}}$

ie wait until declining rate of growth of wine just equals the constant int rate on cash amt

We must check SOC

$$\frac{d^2A}{dt^2} = \frac{d}{dt}\left(A\left(\tfrac{1}{2}t^{-1/2} - r\right)\right) =$$ use product rule

$$= \frac{dA}{dt}\left(\tfrac{1}{2}t^{-1/2} - r\right) + A\left(-\tfrac{1}{4}t^{-3/2}\right)$$

The 1st term $= 0$, because of the FOC (check)

$$\therefore \frac{d^2A}{dt^2} = \frac{-A}{4t^{3/2}} < 0 \text{ at } t^* > 0$$

given $A > 0$

$\therefore$ t^* is the value which maximise A

Timber Cutting

↳ Do in workshop with Gabi

BTW $\log t = \ln t$ for us, ie if I see log, I assume it is the natural log, ie $\log t = \log_e t = \ln t$

Also NB $\frac{d}{dt}\left(\ln(f(t))\right) = \frac{1}{f(t)} \cdot f'(t)$

Remember, for PV, we may need to consider any costs (besides sunk costs)

Further Applications 10.7

If $y = f(t)$

Instantaneous rate of growth =

$$r_y = \frac{dy/dt}{y} = \frac{f'(t)}{f(t)} = \frac{\text{marginal fn}}{\text{total fn}}.$$

But $\frac{d}{dt}\left(\ln(f(t))\right) = \frac{f'(t)}{f(t)}$

∴ can use either approach to calculate r_y. (ie take $\ln(f(t))$ & differentiate)

eg $V = Ae^{rt}$, what is r

$\ln V = \ln A + \ln e^{rt}$

$\ln V = \ln A + rt$

∴ $r_y = \frac{d}{dt}(\ln A + rt)$

$r_y = r$

eg $y = 4^t$

$\ln y = \ln 4^t = t \ln 4$

∴ $r_y = \frac{d}{dt}(\ln y) = \ln 4$

Rate of Growth of a Combo of fns

$$y = uv \quad \text{where} \quad \begin{cases} u = f(t) \\ v = g(t) \end{cases}$$

$$\ln y = \ln u + \ln v$$

$$r_y = \frac{d}{dt}(\ln y) = \frac{d}{dt}(\ln u) + \frac{d}{dt}(\ln v)$$

$$r_y = r_u + r_v$$

Similarly

$$y = u/v$$

$$r_y = r_u - r_v$$

eg Consumption C, growing at rate α, population H growing at rate B, what is rate of growth of per capital cons?

$\frac{C}{H}$ has rate of growth

$$= \frac{d}{dt}\left(\ln \frac{C}{H}\right) = \frac{d}{dt}(\ln C - \ln H) = r_C - r_H$$
$$= \alpha - B$$

What about $z = u + v$ $\quad u = f(t)$, $v = g(t)$

$$\ln z = \ln(u+v)$$

$$r_z = \frac{d}{dt}\ln z = \frac{d}{dt}\ln(u+v)$$

$$= \frac{u' + v'}{u+v} = \frac{f' + g'}{u+v}$$

We know $r_u = \dfrac{f'(t)}{f(t)}$ (1st rule)

$$\therefore f'(t) = r_u . f(t)$$

$$\& \; g'(t) = r_v \, g(t)$$

$$\therefore \frac{u . r_u + v . r_v}{u+v} = \frac{u}{u+v} r_u + \frac{v}{u+v} r_v$$

$\therefore$ rate of growth of a sum is the weighted avg of the rates of growth of the components

similarly

$$r_{(u-v)} = \frac{u}{u-v} r_u - \frac{v}{u-v} r_v \quad \text{(Check)}$$

eg Exports goods $q = q(t)$ has growth rate $= a/t$

Export Services $S = s(t)$ has growth rate $= b/t$

What is growth rate of total exports? $X(t) = q(t) + s(t)$

$$\therefore r_X = \frac{q}{q+s} r_q + \frac{s}{q+s} r_s$$

$$= \frac{q}{X}\left(\frac{a}{t}\right) + \frac{s}{X}\left(\frac{b}{t}\right)$$

$$= \frac{qa + sb}{Xt}.$$

Point Elasticity

given $y = f(x)$

We know $\varepsilon = \dfrac{\text{marginal}}{\text{avg}}$

$$= \frac{f'(x)}{f(x)/x} = \frac{dy}{dx} \cdot \frac{x}{y}$$

Now lets try with our new tools

$y = f(x)$

What if we differentiate $\ln y$ w.r.t $\ln x$?

Let $u \equiv \ln y$

$v \equiv \ln x$

$\therefore$ $u = \ln y$ $\quad y = f(x)$ $\quad x = e^{\ln x} = e^{v}$

$\therefore$ diff w.r.t $\ln x$

$$\frac{d(\ln y)}{d(\ln x)} = \frac{du}{dv} = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv}$$

$$= \frac{1}{y} \cdot \frac{dy}{dx} \cdot e^{v}$$

$$= \frac{1}{y} \cdot \frac{dy}{dx} \cdot x$$

$$= \frac{dy}{dx} \cdot \frac{x}{y}$$

This is what we calculated for the pt elasticity of $y = f(x)$.

$$\therefore \quad \varepsilon_{yx} = \frac{d(\ln y)}{d(\ln x)}$$

This method might be easier depending on the functional form of y.

eg Given $= \frac{k}{P}$, k positive consta

This is a hyperbola, & has unit elasticity for all P. Let's show this.

$$Q = \frac{k}{P}$$

$$\ln Q = \ln k - \ln P$$

$$\varepsilon_D = \frac{d(\ln Q)}{d(\ln P)} = -1 \therefore |\varepsilon_d| = 1$$

Can also find this using the usual formula $\varepsilon = \frac{\text{marginal}}{\text{avg}}$

Check you can show:

$$\varepsilon_{yx} = \varepsilon_{yw}\,\varepsilon_{wx}$$

if $y = g(w)$ & $w = h(x)$

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