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ECO 4112 F

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[CHAPTER 05]

Chapter 5

(1)

Revise
Determinants
Nec & Suff.

P 82 → 121

Lecture 2

How do we know if A^{-1} exists?

Conditions for Non Singularity

A must be square = necessary

Necessary = prerequisite

$$\boxed{p \Rightarrow q}$$

if p, then q (if p, then had to have had q)

p implies q

p only if q

a – person is father

b – person is male

$a \Rightarrow b$ does $b \Rightarrow a$

not necessarily.

Sufficient

q is sufficient for p

$$p \Leftarrow q$$

p = can get to us

q = flies to us

if we see q, we def see p

but we can see p without q – not necessa

2

q both nec & suff

$p \Leftrightarrow q$.

Is a necessary condition sufficient? need o be male to be a father (necessary) but is it sufficient?

Is a sufficient condition necessary?

eg A $\Rightarrow$ passing course

A is sufficient for passing

but B also sufficient (A is not necessary)

Necessary & sufficient
└ best └ guarentee

eg male & your baby mama is preg with your child $\Leftrightarrow$ father.

Conditions for Nonsingularity (NS)

Squareness = necessary for NS

sufficient = LI rows or cols.

$\boxed{NS} \iff \boxed{\text{square } \& \text{ LI}}$

$$n \times n \; A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix}$$

where v_1' = row vector
$= [a_{11} \; a_{12} \cdots a_{1n}]$

k_i which

$\not\exists$ $\quad {}_i v_i' = \underset{1\times n}{0}$

eg $\begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 8 & 10 \end{bmatrix} \quad \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix}$

$\therefore \; 2v_1' \quad v_3' + 0v_2'$

or $\quad 2v_1' - v_3' + 0v_2' = 0$
$= [0 \; 0 \; 0]$

LD

Can we see LD easily?
No. 2 ways to find it.
1 echelon mxes (rank)
2 determinant.

First though, why squareness & LI together?

for n unknowns, need n EQNS for unique soln^why? BUT, could have LD = inf solns
OR inconsistent = no soln

eg $\begin{bmatrix} 10 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

$\boxed{\text{L.D}}$

either $d_1 = 2d_2$ inf solns OR $d_1 \neq 2d_2 \Rightarrow$ inconsistent

$\boxed{\text{HW}}$ – check this out

need square & LD

if rows cos are LI

& A is square $\Rightarrow$ NS

$\Rightarrow$ A^{-1} exists

$\Rightarrow$ UNIQUE Soln

$$\boxed{\begin{aligned} Ax &= d \\ x^* &= A^{-1}d \end{aligned}}$$

<u>Rank</u>

<u>RI</u> only for square?

No. Any $m \times n$

Max no of LI rows for $m \times n$ mx $= r$ then rank$(A) =$

<u>Rank</u> = max no LI rows/cos

$\leq$ m n whichever is smaller.

if 2×2, can see it.

But otherwise need something else. Need to transform system in OK way which allows to see it.

(6)

Use <u>row operations</u> to transfo

(1) swap rows
(2) $\times$ row by $k \neq 0$
(3) add $k \times$ row to another row

(<u>why?</u> same as solving & substitution)

These dont alter rank of mx.

<u>EG</u> What is RE form?

$$\begin{bmatrix} 1 & - & - \\ 0 & 1 & - \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & - & - & - \\ 0 & 0 & 1 & - \end{bmatrix}$$

$$\begin{bmatrix} 1 & - & - \\ 0 & 1 & - \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(1) non zero rows above zero rows
(2) 1st non zero element =
(3) must be zeroes below 1st non zero element of a row.

P86

(7)

<u>eg</u>

$$A = \begin{bmatrix} 0 & -11 & -4 \\ 2 & 6 & 2 \\ 4 & 1 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 6 & 2 \\ 0 & -11 & -4 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 2 & 6 & 2 \\ 0 & -11 & -4 \end{bmatrix} \qquad \tfrac{1}{4} R_1$$

$$A_3 \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & 5\frac{1}{2} & 2 \\ 0 & -11 & -4 \end{bmatrix} \qquad R_2 - 2R_1$$

$5\frac{1}{2} = \frac{11}{2}$

$$A_4 \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{4}{11} \\ 0 & -11 & -4 \end{bmatrix} \qquad R_2 \times \frac{2}{11}$$

$$A_5 \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{4}{11} \\ 0 & 0 & 0 \end{bmatrix} \qquad R_3 + 11 \times R_2$$

$\therefore$ 2 LI rows

$\therefore$ rank $A = 2$

Is there 1 unique way of getting row echelon form?

Can we do echelon mx transformations for non square matrices?

<u>For a square mx.</u>

to be NS, • it must have n LI rows or cols.

- it must have rank n
- echelon mx must have n nonzero rows.

eg 3X3 mx with rank(A)=2

∴ A is singular.

Determinants to test for NS

Row echelon is one method.

det(A) = |A| is another

|A| = unique scalar.

= defined only for square mxes

$|A|$ where A is 1×1 = a_{11}

eg $|[3]| = 3$

|A| is <u>not</u> absolute value

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $|A| = ad - bc$

eg $\left|\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right| = 4 - 6 = -2$

If A has LI rows, then

$|A| = 0$. $\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$ $|A| = 16 - 16 = 0$

3 order Determinant

eg $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Expand by any row or col:

1st row =

$a_{11} \begin{vmatrix} & \end{vmatrix} - a_{12} \begin{vmatrix} & \end{vmatrix} + a_{13} \begin{vmatrix} & \end{vmatrix}$

- how to find signs?
- What goes in the brackets $| \; |$?

Signs : if subscripts add to
add number, sign < 0
even number, sign > 0

eg a_{13} $1+3 = 4$ $\therefore$ sign > 0
a_{23} $2+3 = 5$ sign < 0

What goes in the | |'s?

for a_{11}, block out row & col which contain a_{11}

eg $\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{bmatrix}$

these determinants are called the minors of a_{11}, a_{12} & a_{13}

$$\therefore \quad a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

NB
Can expand by any row | col & get same answer

HW – check this.

<u>Minor</u> = |mx| associated with each element

eg $M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

ign

$(\ 1)^{i+j} |M_{ij}|$

$i+j$ is even.

<u>eg</u> $\Big| \qquad \Big|$

M

C_3

$\therefore$ 3X3 mx has determina

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{1}$$

$$= a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}$$

$$= \sum_{j=1}^{3} a_{1j}|C_{1j}|$$

or with 2nd row

$$= \sum_{j=1}^{3} a_{2j}|C_{2j}|$$

OR with 3rd col

$$= \sum_{i=1}^{3} a_{i3}|C_{i3}|$$

$\boxed{\text{HW}}$ Try this with example for 4X4 mx B

$$|B| = \sum_{j=1}^{4} b_{1j} |C_{1j}| \quad \text{1st row}$$

$$= \sum_{i=1}^{4} b_{i2} |C_{i2}| \quad \text{2nd col}$$

This is called Laplace Expansion.

eg $|A| = \begin{vmatrix} 5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0 \end{vmatrix}$

3rd col $= 1\begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} - 0\begin{vmatrix} 5 & 6 \\ 7 & -3 \end{vmatrix} + 0\begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix}$

$= -6 - 21$

$= -27$

NB Look for cols/rows with zeroes.

$$|A| = \sum_{j=1}^{n} a_{ij} |C_{ij}| \quad \text{ith row}$$

$$= \sum_{i=1}^{n} a_{ij} |C_{ij}| \quad \text{jth col}$$

Properties of Determinants

1. Swapping rows / cols does not affect $|A|$
   ie $|A^T| = |A|$

2. Swapping any 2 rows/cols changes sign of $|A|$

3. X a row/col by k yields $k|A|$

$\boxed{\text{HW}}$ – check these 3 properties

(4) Adding a multiple of a row | col to another row | col doesnt change $|A|$

(5) if 2 rows | cols are multiples of each other, $|A| = 0$

Why? Because getting such an A into row echelon form gives a row/col of zeroes, thus $|A| = 0$

Criterion for a mx to be NS

eg $Ax = d$

Unique Soln iff rows of A are LI

ie A is N.S

But if $R_1 = 2R_2$ (for eg) rows are L.D & no unique soln

But if rows are L.D
we have a row of zeroe
$\therefore$ $|A| = 0$.
$\therefore$ for $Ax = d$ where $A = n \times n$ mx,

$|A| \neq 0 \iff$ rows/cols are L.I.
$\iff$ A is N.S
$\iff$ A^{-1} exists
$\iff$ a unique soln exists $x^* = A^{-1}d$.

- This is handy. Why?
- Are values x^* necessarily admissable values?

NB
why row & col LI?
P96

RANK (again)

rank A = max num of LI rows
= max order of a non-zero determinant from the rows & cols of a mx

We know $r(A) \leq \min\{m, n\}$
for $A_{m \times n}$

For product AB
$r(AB) \leq \min\{r(A), r(B)\}$
Why? P98 proof – see next page

Finding the Inverse

If A is NS, then A^{-1} exists, as does unique soln $x^* = A^{-1}c$

How to find A^{-1}, if $|A| \neq 0$?

Property 6 of $|A|$ The expansion of a determinant by alien cofactors (c_{ij} of wrong row/col) is always zero.

P18 Proof (P98)

Given $R(AB) \leq \min\{r(A), r(B)\}$

Given mx A, with $R(A) = j$

& multiply by NS mx B,

then prove $r(AB) = j$

2 parts

Part 1: Consider 3 cases:

1. $r(A) < r(B)$
2. $r(A) = r(B)$
3. $r(A) > r(B)$

Given either 1 or 2, then

$r(AB) \leq r(A) = j$

Given 3

$r(AB) \leq r(B) < r(A) = j$

$$\boxed{r(AB) \leq r(A) = j}$$

Part 2

Consider $(AB)B^{-1} = A$

$\therefore r((AB)B^{-1}) \leq \min\{r(AB), r(B^{-1})\}$

$\therefore$ We can say:

$$r((AB)B^{-1}) \leq r(AB)$$

Simplify

$$r(A) \leq r(AB)$$

$$j \leq r(AB)$$

$\therefore$ from part 1 & 2 combined

$$\underbrace{j \leq r(AB)}_{\text{part 1}} \qquad \underbrace{r(AB) \leq j}_{\text{Part 2}}$$

$\therefore r(AB) =$

$\therefore r(AB) = j$'s roven.

Section 5.4

eg
$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & 2 & 4 \\ 0 & 1 & 2 \end{vmatrix}$$

$= a_{11}|C_{21}| + a_{12}|C_{22}| + a_{13}|C_{23}$

= 1st row with 2nd row $|C_{ij}|$

$= 1.|C_{21}| + 2|C_{22}| + 0|C_{23}|$

$= -1\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$

$= -1(4) + 2(2) - 0(1)$

$= 0$

eg $[A] = [a_{ij}]$

We know (1st row elements, 2nd row cofactors)

$\sum_{j=1}^{3} a_{1j}|C_{2j}|$ (Note mistake in subscripts p99)

$= a_{11}|C_{21}| + a_{12}|C_{22}| + a_{13}|C_{23}|$

$= 0$ Why? Expand & Check

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How else do we know

$$\sum a_{1j} |C_{2j}| = 0 \; ?$$

This is the same as finding the determinant of this mx:

$$|A^*| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{(Check)}$$

But $|A^*|$ is same as $|A|$, except 2nd & 1st rows are identical, ∴ LD ∴ $|A^*| = 0$

∴ expansion by alien cofactors yields zero, for any row/col.

(5.13)

$$\sum a_{ij} |C_{i'j}| = 0 \quad (i \neq i') \text{ ROW}$$

$$\sum a_{ij} |C_{ij'}| = 0 \quad (j \neq j') \text{ COL}$$

Matrix Inversion

Assume $A_{n \times n}$ NS mx.

$$\underset{n\times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad |A| \neq 0$$

We can form a matrix of cofactors as follows:

$$\underset{n\times n}{C} = \text{adj}\, A = \begin{bmatrix} |C_{11}| & |C_{12}| & \dots & |C_{n1}| \\ |C_{21}| & |C_{22}| & \dots & |C_{2n}| \\ \vdots & & & \\ |C_{n1}| & |C_{n2}| & \dots & |C_{nn}| \end{bmatrix}$$

$$C' = \text{adj}\, A = \begin{bmatrix} |C_{11}| & |C_{21}| & \dots & |C_{n1}| \\ |C_{12}| & & & \\ \vdots & & & \\ |C_{1n}| & |C_{2n}| & \dots & |C_{nn}| \end{bmatrix}$$

(adjoint of A)

$AC' =$ nxn mx

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$$AC' = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ & & & \\ |C_{1n}| & \cdots & & |C_{nn}| \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{n} a_{1j} |C_{1j}| & \\ & \\ & \end{bmatrix}$$

why do the diagonal element resolve to zero? Because of S.8, S.13 - expansion by alient cofactors defn of a determinant

$$AC' = adj(A) = \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & \cdots & 0 \\ 0 & 1 & \vdots \\ \vdots & & \\ 0 & \cdots 0 & 1 \end{bmatrix} = |A| . I_n$$

$\div$ by $|A|$

$$\frac{AC'}{|A|} = I_n$$

preX by A^{-1}

$$\frac{A^{-1}AC'}{|A|} = A^{-1}$$

$$A^{-1} = \frac{1}{|A|} . C' = \frac{1}{|A|} . adj\, A$$

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$\therefore$ to find A^{-1}

1. find $|A|$
2. find $\operatorname{adj} A = \begin{bmatrix} \text{cofactors} \\ \text{of } A \end{bmatrix}^T$
3. find $A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj} A$

eg $\begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix} = A$. find A^{-1}

$|A| = 99 \neq 0$.

$$\operatorname{adj} A = \begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} \end{bmatrix}^T$$

* dont forget signs or to take the transpose

$$= \begin{bmatrix} 21 & 6 & -9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix}$$

$$\therefore \quad B^{-1} = \frac{1}{99} \operatorname{adj} A$$

Cramer's Rule

$$Ax = d \qquad A = n \times n$$

$$\therefore \quad x^* = A^{-1} d = \frac{1}{|A|} \operatorname{adj} A \cdot d$$

$$\therefore \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix} = \frac{1}{|A|} \underset{n \times n}{\begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \vdots & & & \\ |C_{1n}| & & \cdots & |C_{nn}| \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}}$$

$$= \frac{1}{|A|} \underset{n \times 1}{\begin{bmatrix} |C_{11}|d_1 + |C_{21}|d_2 + \cdots |C_{n1}|d_n \\ |C_{12}|d_1 + |C_{22}|d_2 + \cdots |C_{n2}|d_n \\ \vdots \\ |C_{1n}|d_1 + \cdots + |C_{nn}|d_n \end{bmatrix}}$$

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$$x^* = \frac{1}{|A|} \begin{bmatrix} \sum_{i=1}^{n} |C_{i1}| d_i \\ \sum_{i=1}^{n} |C_{i2}| d_i \\ \vdots \\ \sum_{i=1}^{n} |C_{in}| d_i \end{bmatrix}_{n \times 1}$$

$$x_1^* = \frac{1}{|A|} \cdot \sum_{i=1}^{n} |C_{i1}| d_i$$

$$x_2^* = \frac{1}{|A|} \sum_{i=1}^{n} |C_{i2}| d_i$$

$$x_n^* = \frac{1}{|A|} \sum_{i=1}^{n} |C_{in}| d_i$$

What are these terms?

$$x_1^* = \frac{1}{|A|} \cdot |A_1|$$ what is $|A_1|$?

$$|A_1| = \begin{vmatrix} d_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ d_2 & a_{22} & & & \\ \vdots & \vdots & & & \\ d_n & a_{n2} & \cdots & & a_{nn} \end{vmatrix} = d_1|C_{11}| + d_2|C_{21}| + \cdots d_n|C_{n1}| = \sum_{i=1}^{n} d_i |C_{i1}|$$

ie $|A_1| = |A|$ but with 1st col replaced by d,
& $|A_1|$ calculated by expanding by 1st col.

Similarly $x_2^* = \frac{1}{|A|}|A_2|$

$|A_2| = |A|$ but with col 2 replaced by d.

etc.

$$\therefore \; x_j^* = \frac{1}{|A|} \cdot |A_j|$$

$$= \frac{1}{|A|} \begin{vmatrix} a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & & d_2 & \cdots & a_{2n} \\ \vdots & & & \vdots & & \\ a_{n1} & a_{n2} & & d_n & \cdots & a_{nn} \end{vmatrix}$$

jth col replaced by d

<u>eg</u> Solve $5x_1 + 3x_2 = 30$
$6x_1 - 2x_2 = 8$

$$\begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 30 \\ 8 \end{bmatrix}$$

$$\therefore \quad x_1 = \frac{1}{|A|} \cdot |A_1|$$

$$= \frac{1}{-28} \begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix}$$

$$= \frac{1}{-28} \cdot -84$$

$x_1^* = 3$ Similarly $x_2^* = 5$

eg P105

Homogenous System Equations

if $d = 0$

ie $Ax = 0$ ie $\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

= homogenous system

if A is NS, one solution is $x^* = 0$

Because $x^* = \underset{n\times n}{A^{-1}} \underset{n\times 1}{d} = A^{-1} \cdot 0 = \underset{n\times 1}{0}$

Why? $x_j^* = \frac{1}{|A|} \cdot |A_j|$

but A_j has a column of zeroes, $\therefore |A_j| = 0$

Only if $|A| = 0$ ie A is 'S',
then $x_j^* = \frac{|A_j|}{|A|} = \frac{0}{0}$ = undefined

∴ no unique soln.
Why infinitely many solns?
P106 – work it out.

$Ax = d$

	$d \neq 0$	$d = 0$
$\|A\| \neq 0$ A is NS	UNIQUE SOLN $x^* \neq 0$ $x^* = \frac{1}{\|A\|} \text{adj} A . d$	Unique Soln $x^* = 0$
$\|A\| = 0$ A is S	Infinite # of solns	Infinite solns & $x^* = 0$
Inconsistent EQNS	No soln	Not possible (why?)

Market & National Income Models

Please work through examples on p107 (market model) & P108 (NI model)

ISLM

2 sectors - real goods & monetary

GOODS MKT

P109

$$Y = C + I + G$$
$$C = a + b(1-t)Y$$
$$I = d - ei$$
$$G = G_0$$

Y, C, I, i = endogenous

G = exog

$a\ d\ e\ b\ t$ = parameters

MONEY MKT

$$M_d = M_s$$
$$M_d = kY - li$$
$$M_s = M_0$$

M_0 = exog

k, l = parameters

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<u>money mkt</u>

$$M_0 = kY - li$$

<u>All together:</u>

$$Y - C - I = G_0$$
$$b(1-t)Y - C = -a$$
$$I + ei = d$$
$$kY - li = M_0$$

In mx format

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ b(1-t) & -1 & 0 & 0 \\ 0 & 0 & 1 & e \\ k & 0 & 0 & -l \end{bmatrix} \begin{bmatrix} Y \\ C \\ I \\ i \end{bmatrix} = \begin{bmatrix} G_0 \\ -a \\ d \\ M_0 \end{bmatrix}.$$

$|A|$ = using 3rd row

$$= 0|C_{31}| - 0|C_{32}| + |C_{33}| - e|C_{34}|$$

$$= \begin{vmatrix} 1 & -1 & 0 \\ b(1-t) & -1 & 0 \\ k & 0 & -l \end{vmatrix} - e \begin{vmatrix} 1 & -1 & -1 \\ b(1-t) & -1 & 0 \\ k & 0 & 0 \end{vmatrix}$$

$$= -\ell(-1 + b(1-t)) - e(k(-1))$$

$$= \ell - b\ell(1-t) + ke$$

$$= \ell(1 - b(1-t)) + ke$$

How do we know if $|A| \neq 0$

Need assns.

Assume $0 \leq t \leq 1$ $t =$ tax rate

$0 \leq b \leq 1$ $b =$ mpc

$k, e > 0$

$\ell > 0$

$\therefore |A| > 0$.

Now, solve for C^*.

$$C^* = \frac{1}{|A|} \cdot |A_2|$$

$$= \frac{1}{|A|} \left| \begin{bmatrix} 1 & G_0 & -1 & 0 \\ b(1-t) & -a & 0 & 0 \\ 0 & d & 1 & e \\ k & M_0 & 0 & -\ell \end{bmatrix} \right|$$

expand by 2nd row, or 4th/3rd cols

etc

Why mx algebra, & not substitution?

- compact
- can test for existence of a soln.
- used in Econometrics from hereon out — v useful.

Leontief Input Output Models

- n industries
- What level of output should each of the n industries produce, to satisfy total demand for each of the n products?
- Interindustry dependence
- Dont want bottlenecks

Assns

1. 1 homogenous product per industr
2. fixed input ratio per industry
3. Constant returns to scale.

Unrealistic?

Output

Input

$$\begin{array}{cc} & \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & & \text{N} \end{array} \\ \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \\ \text{N} \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & & a_{2n} \\ & & & & \\ & & & & \\ a_{n1} & \cdots & & \cdots & a_{nn} \end{bmatrix} \end{array}$$

$= A$

a_{ij} = amount of i required to produce 1\$ of j.

To produce 1 unit of j, I need a_{1j} of good 1, a_{2j} of good 2, ... & a_{nj} of good n.

- Prices are given
- a_{ij} = input coefficient
- What is a_{32}? What is the sum of the fourth column?

Open Model

Final Demand = input DD (interindustri-) + consumer, govt & foreign DD

All inputs = intermediate inputs

+ primary inputs (labour etc)

We include an open sector = govt, cons, foreign

$\therefore$ sum each column in $A < 1$

= partial input cost (not including primary inputs < 1

What if $\sum_{i=1}^{n} a_{ij}$ $(j = 1, 2, \dots n) > 1$?

How much do we pay the primary inputs? $1 - \sum_{i=1}^{n} a_{ij}$

$1 -$ column sum

To meet demand, we need:

$$x_1 = \underbrace{a_{11}x_1}_{\text{input demand}} + \underbrace{a_{12}x_2}_{\text{input DD from industry 2 for good 1.}} + \dots + a_{1n}x_n + \underbrace{d_1}_{\text{final DD}}$$

etc etc

$$x_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + d_n$$

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Move all terms to

$$(1-a_{11})x_1 - a_{12}x_2 - \cdots - a_{1n}x = d_1$$

$$-a_{21}x_1 + (1-a_{22})x_2 - \cdots - a_{2n}x = d_2$$

$$\vdots$$

$$-a_{n1}x_1 - a_n \cdots + (1-a_{nn})x_n = d_n$$

Mx

$$\begin{bmatrix} (1-a_{11}) & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & (1-a_{22}) & \cdots & -a_{2n} \\ \vdots & & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & (1-a_{nn}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

Ignoring 1s, we have $-A = [-a_{ij}]$

This mx is $I - A$ (Check)

$$(I-A)x = d$$

$$x = (I-A)^{-1}d$$

(35)

$I - A$ = Leontief Matrix.

Example P115

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}$$

Each col sum < 1 ✓

a_{0j} = \$ amt of primary input

$a_{01} = 1 - 0.7 = 0.3$

$a_{02} = 0.3$

$a_{03} = 0.4$

$(I - A)x = d$

$$\begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d \\ d \end{bmatrix}$$

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = (I - A)^{-1} = \frac{1}{0.384} \begin{bmatrix} 0 & & \\ 0.34 & 0 & \\ 0 & & \end{bmatrix}$$

H

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If $d = \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$ billion dollars,

Show $x_1^* = 24.84$

$x_2^* = 20.68$

$x_3^* = 18.36$

This is done using some amt of primary input. What if the amt is not available?

$$\sum_{j=1}^{3} a_{0j} x_j^* = 0.3(24.84) + 0.3(20.68) + 0.4(18.36) = 21 \text{ billion}$$

If we dont have 21 b\$ worth of primary input, we must revise production downwards

- Using inverse method, changing d & then finding x^* is easy, unlike Cramer's rule

Non Negative Solutions

if $(I-A)$ is NS, we can find x^*. What if x^* arent all non negative?

This is not going to be the case all the time. When will it be?

Hawkins & Simon

Given (a) $n \times n$ mx B, with $b_{ij} \leq 0$ $(i \neq j)$ ie off diagonal elements ≤ 0

and (b) an $n \times 1$ vector $d \geq 0$,

there exists an $n \times 1$ vector $x^* \geq 0$, s.t $Bx^* = d$, iff

$|B_m| > 0 \quad (m = 1, 2, \ldots n)$

ie only if leading principal minors of B are positive.

So (a) $(I-A) = B$, all $b_{i} \leq 0$, $j \neq i$

(b) $d \geq 0$

(38)

What are leading PMs?

PM = principal minor

A kth order PM is obtained by deleting any $n-k$ rows & the same $n-k$ columns & then taking the determinant

eg $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

3rd order PM : delete $3-3$ rows & cc ie entire matrix

$$|A_3| = |A|$$

2nd order PM : delete any $3-2 = 1$ rows | cols

ie $|A_2|$ will be 3 2X2 mxes

$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ $\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$ $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

delete 1st row|col — 2nd row|col — 3rd row|col

1st order : delete $3-1 = 2$ rows | cols

39 $|A_1|$

1st order PMs

$= |a_{33}|, \; |a_{11}|, \; |a_{22}|$

delete 1st & 2nd rows/cols

delete 2nd & 3rd rows/cols

delete 1st & 3rd rows/cols

Now, what are <u>leading</u> PMs?

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$|A_1| = |a_{11}|$

$|A_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

etc etc

<u>NB</u>: Notation

$|A_2|$ is a 2X2 PM

$|A_3|$ is a 3X3 PM

etc

Why <u>principal</u> minors?

Because the diagonal elements of all the minors above are the principal diagonal elements of A.

- So a <u>minor</u> $|M_{ij}|$ is the determinant of the matrix obtained by deleting the ith row & jth column
- A <u>principal minor</u> $|A_k|$ is the det of the mx obtained by deleting <u>any</u> n−k rows & cols (same cols as rows)
- A <u>leading</u> principal minor $|A_m|$ is obtained by taking the 1st m principal diagonal elements in $|A|$ & their corresponding off-diagonal element (Remember the pattern 

<u>Back to Hawkins-Simon</u>

L Do we prove it?

L We need the leading PMs of $I-A$ to be positive to yield $x^* \geq 0$.

2X2 example

$$I - A = \begin{bmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{bmatrix}$$

We need: (a) $b_{ij} \leq 0, \ i \neq j$

YES

(b) $|B_m| > 0$

leading PMs:

$|B_1| > 0 \Rightarrow 1 - a_{11} > 0$

$a_{11} < 1$

$|B_2| > 0 \Rightarrow (1-a_{11})(1-a_{22}) - a_{12}a_{21}$

> 0

$$\Rightarrow a_{11} + a_{12}a_{21} + \underbrace{(1-a_{11})a_{22}}_{+ve} < 1$$

$$\underset{\text{direct}}{\underset{\uparrow}{a_{11}}} + \underset{\text{indirect}}{\underset{\uparrow}{a_{12}a_{21}}} < 1.$$

uses of (1)

If the conditions are met, we get meaningful results for x^*

Closed Model

If we absorb the exogenous sector into the system as another industry, it becomes a closed model.

- No final DD or primary input
- All goods now intermediate
- $n+1$ industries.
- We still assume a fixed input ratio — HHs consume each commodity in fixed propⁿ to labour they supply.
- Now have a homogenous system $d = 0$
- will have a soln iff $I - A$ has $\det(I-A) = 0$, because $I - A$ is singular. Why?

- See p106 for answer to previous question
- Why is $\det(I-A)=0$ in closed system?

$$\begin{bmatrix} 1-a_{00} & -a_{01} & -a_{02} & -a_{03} \\ -a_{10} & 1-a_{11} & -a_{12} & -a_{13} \\ -a_{20} & -a_{21} & 1-a_{22} & -a_{23} \\ -a_{30} & -a_{31} & -a_{32} & 1-a_{33} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

x_0 = used to be primary input (input requirement for industries 1 through n now

- No final DD, all goods intermediate
- Each column sum must equal 1 : $a_{0j} + a_{1j} + a_{2j} + a_{3j} = 1$

OR $a_{0j} = 1 - a_{1j} - a_{2j} - a_{3j}$

∴ in every column of $I-A$, the top row element $= 1 -$ other col element

∴ 4 rows L.D

∴ $|I-A| = 0$

have nontrivial solutions,
Infinite number
∴ no 1 correct answer.

Static Analysis

- ignores how we get to EQM values, & how long it takes to get there (might not be relevant any more if exog vars change)
- EQM state may be unattainable if it is unstable.

Comparative Statics

= shifts of EQM state w.r.t exog var changes

Dynamic Analysis:

Attainability / Stability of EQM