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ECO 4112 F

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[CHAPTER 04]

Math fo Economists

① Economi theory exog? endog?
⇒ models equation
⇒ General/Partial systems, either way, many equations
How to solve linear sets of EQNS — linear algebra.
Is there a soln? How many?
= STATIC ANALYSIS
What if exog vars shift? Do eqm values change? How?
= COMPARATIVE STATICS (derivatives
What if cant hold vars constant
Total Differentiation
What if cant solve for endog fn(exog
IFT to see derivative in any case
Are all equilibria good? How to tell? OPTIMISATION
Remember, $\leq$. (only equality)
What about time? How do

variables change with time?
How long to get to EQM?
Will we stay there?
= <u>DYNAMICS</u>
But, first we need tool
= <u>INTEGRATION</u>,
rules logs, exponents
<u>Why does math help?</u>
why not just word? Graphs?
Crarifies assumptions, relnship
relative size of effects
More than 3 dimensions
ealth math theory to help
Connection to Econometrics?
metr
=
=
=

Abbreviations

LI = linearly independent

LD = linear dependence

mx = matrix

HW = homework

def = definitely

defn = definition

$A \iff B$ = A if and only if B
A iff B.

$A \Rightarrow B$ = A implies B
B is necessary for A
A is sufficient for B

$\forall$ = for all

Soln = solution

col = column

NS = non singular

Abbreviations Continued

Vars = variables

pt = point

x = column vector

x' = row vector

Mult = multiplication

$u \cdot u$ = inner product = scalar

$= \sum_{i=1}^{n} u_i^2$

Combo = combination

LC = linear combination

dist = distance

fn = function

Comm = commutative

Assoc = associative

Prob = probability

popn = population

Lecture 1

Why matrices?

P48

- 81

33 pages

Please also revise mx addition sub scalar x x
vector PS9
PS1

- 3x3 & more systems
- compact
- can test for soln without solving (using determinant)
- can find soln
- used a LOT

But only for $\boxed{\text{LINEAR}}$ eqns.

how realistic? Often a good approximation.

or can transform:

eg
$$y = ax^b$$

$$\log y = \log a + b \log x$$

linear in $\log x$, $\log y$

Matrices 2 ors

c_1 =

=

x_1

1 x

x_1 + + . . . dm

a_i

row

etc

ors

dimension = no row & cols

$m \times n$ = m row

③

Alter dimensions, get special mxes

eg $n \times n$ = square $\begin{array}{cccc} & & & 1 \\ \multicolumn{3}{c}{\begin{bmatrix} & & \\ & & \end{bmatrix}} & \begin{array}{c}2\\3\end{array} \\ 1 & 2 & 3 & \end{array}$

eg $m \times 1$ col vector $\begin{array}{cc} \begin{array}{c}1\\2\\3\\ \vdots \\ m\end{array} & \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \\ & 1 \end{array}$

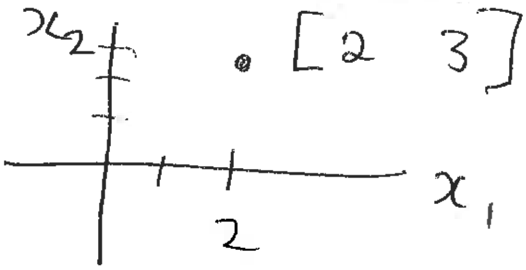
x — col vector

x' — row vector

eg $1 \times n$ row vector $\begin{array}{ccccc} 1 & \multicolumn{4}{c}{\begin{bmatrix} & & & \end{bmatrix}} \\ & 1 & 2 & \cdots & n \end{array}$

Vector = n tuple
= pt in n dimensional space.

eg $\begin{array}{cc} [\,2 & 3\,] \\ x_1 & x_2 \end{array}$



Now express

$Ax = d$ — how to multiply mxes?
— what does equality mean.

Matrix Operations

4

Revision only.

equality: $A = B$

iff – have same dimension

& $a_{ij} = b_{ij}$

for all i, j

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \therefore \quad \begin{matrix} x = 2 \\ y = 3 \end{matrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \therefore \quad \_\_\_\_\_\_$$

Addition & Subtraction

└ same dimensions

eg 2×2

$$[a_{ij}] + [b_{ij}] = [c_{ij}] \quad c_{ij} = a_{ij} + b_{ij}$$

└ same for subtraction

Scalar Mult

$$k[a_{ij}] = [ka_{ij}]$$

every element.

Multiplication

5

L $A \times B = C$

$m \times \underline{n}$ $\underline{n} \times k$ $m \times k$

L some mxes cant be multiplied.

With Vectors

$A \times B = C$

$m \times n$ $n \times p.$ $m \times p.$

mx X = each row A
X each col B
& elements added

eg $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = 2 \times 3$

2×2 2×3

$$= \begin{bmatrix} \underline{a_{11}b_{11} + a_{12}b_{21}} & \underline{a_{11}b_{12} + a_{12}b_{22}} & \underline{a_{11}b_{13} + a_{12}b_{23}} \\ \underline{a_{21}b_{11} + a_{22}b_{21}} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{2} a_{1k}b_{k1} & \sum_{k=1}^{2} a_{1k}b_{k2} & \sum_{k=1}^{2} a_{1k}b_{k3} \\ \sum_{k=1}^{2} a_{2k}b_{k} & \sum_{k=1}^{2} a_{2k}b_{k2} & \sum_{k=1}^{2} a_{2k}b_{k3} \end{bmatrix} \quad 2 \times 3$$

(6)

$$C_{i \times j} = C_{2\times 3} = c_{ij}$$

eg $$c_{11} = \sum_{k=1}^{2} a_{1k} b_{k1} \qquad i=1\ j=1$$

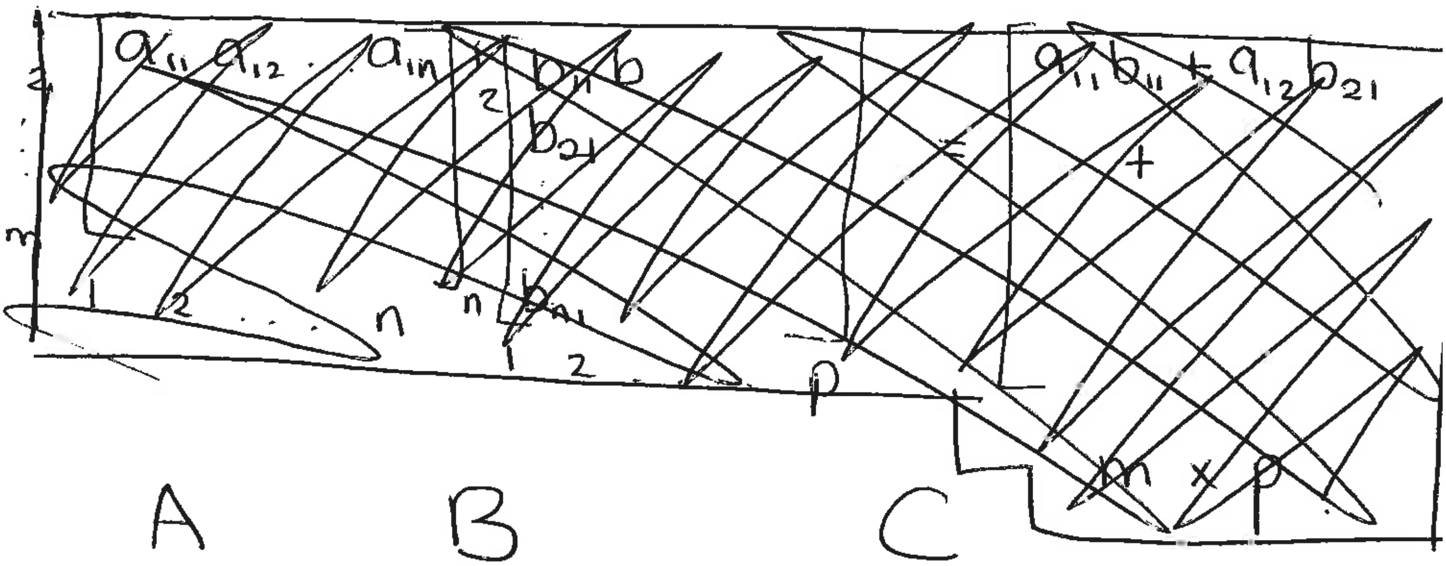
$$c_{13} = \sum_{k=1}^{2} a_{1k} b_{k3} \qquad i=1\ j=3$$

$$c_{22} = \sum_{k=1}^{2} a_{2k} b_{k2} \qquad i=2\ j=2$$

$$c_{23} = \sum_{k=1}^{2} a_{2k} b_{k3} \qquad i=2\ j=3$$

$$\sum_{k=1}^{n} \underbrace{a_{ik}}_{\text{row}} \underbrace{b_{kj}}_{\text{col}} \qquad \begin{matrix} i = 1,2 \\ j = 1,2,3 \end{matrix}$$

=



A B C

2 × 2 2 × 3 2 × 3

$$= \sum_{k=}^{2} a_{ik} \quad {}_{k}^{\,i} \quad ,2,$$

o

C) ^{2}

$$= \sum_{k=}^{n}$$

V

$u = \quad \overline{1} \quad =$

$= \quad\quad =$

$= \quad\quad x.$

1 = not ineq

1 = 1

= $n \times 1$ col

1×1 s

$u = \quad\quad v' = \quad 4$

1×3

2×1

$$uv' = 2 \times 3 = \begin{bmatrix} 3 & 1 & 15 \\ 2 & & 10 \end{bmatrix}$$

1×2

4] 27

V

2 x

$$\underset{1\times n}{u'} = \text{row} = [u_1 \; u_2 \; \dots \; u_n]$$

$$\underset{n\times 1}{u} = \text{col} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$u'u = 1\times 1 = [u_1^2 + u_2^2 + \dots + u_n^2]$$

$$= \left[\sum_{j=1}^{n} u_j^2\right]$$

$$u \cdot u = \sum_{j=1}^{n} u_j^2 = \text{inner product}$$

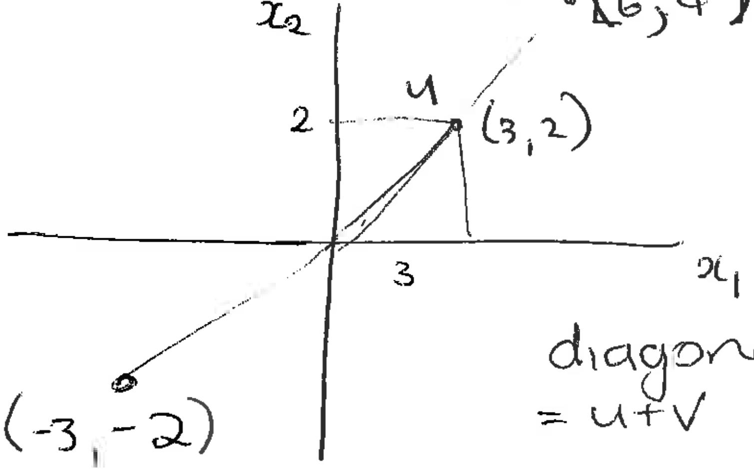
scalar product = row × col

mx = col × row

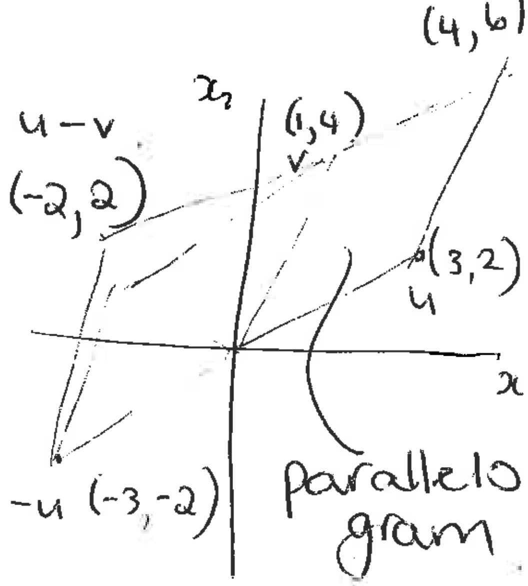
Geometric Interpretation

col, row vector = n vector
= n tuple
= pt in n space

$$u = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad u' = [3 \;\; 2]$$

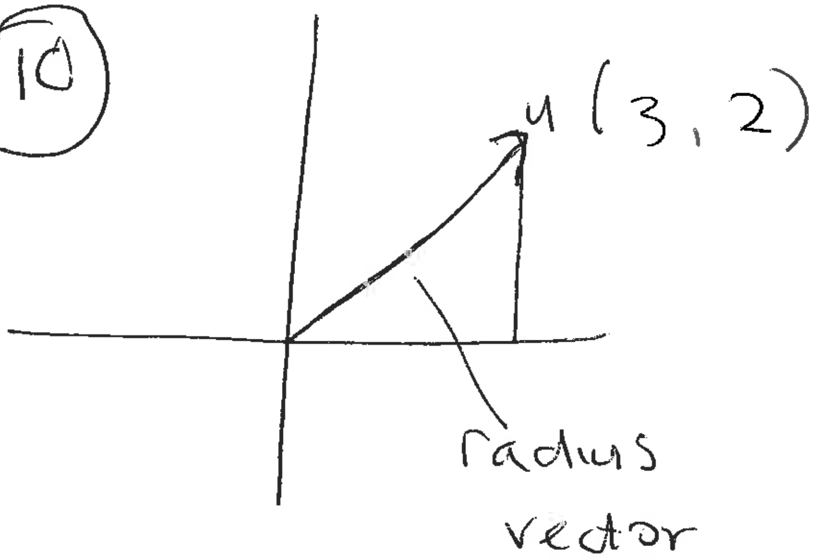


diagonal
= u+V



parallelo
gram

10



- ku
- $k \geq 1$
- $0 < k < 1$
- $k = 0$
- $k < 0$

what happens?

Linear Combo of vectors

eg $3v + 2u = 3\begin{bmatrix}1\\4\end{bmatrix} + 2\begin{bmatrix}3\\2\end{bmatrix}$

$= \begin{bmatrix}3\\12\end{bmatrix} + \begin{bmatrix}6\\4\end{bmatrix}$

$= \begin{bmatrix}9\\16\end{bmatrix}$

$k_1v_1 + k_2v_2 + \dots k_nv_n$

$= \sum_{k=1}^{n} k_i v_i$ $\quad i = 1 \dots n$ $= \boxed{\text{linear combination}} =$ result is a vector

k_i scalar, v_i col

k_i = scalars

v_i = vectors

Linear Dependence

$$\sum_{i=1}^{n} k_i v_i = 0 = \begin{bmatrix}0\\0\end{bmatrix}$$ — zero vector

(11) - if r

2 3 can

as C

eg + 2 3

$V_1 = 2$

- =

$2 - V_3 = 0$

a $= 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

if

t , 2

$V_1 = 2$

- vectors, V_1 ...,

LN

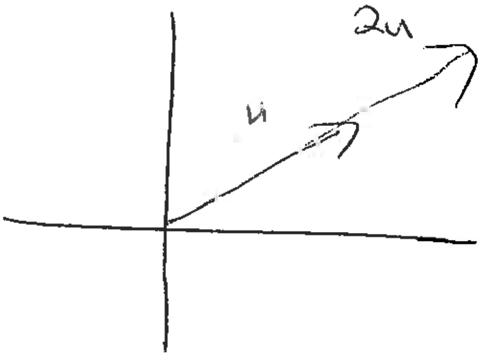
is

s.t

1 1 1

(12) If only k_i which do this are $k_i = 0$ $\forall i$, then LI

Geometrically

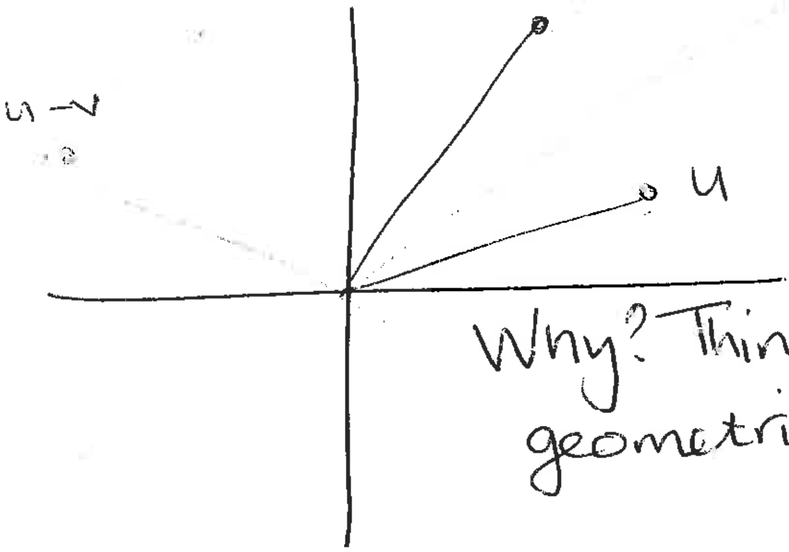


$u, 2u$ obviously LD

$u, -u$ also

u, v – can't do straight line $\therefore$ $u \neq kv$ LI.

If we take 2 LI vectors in 2-space, all other vectors in 2 space can be expressed as a LC of u & v.



Why? Think geometrically

Simplest: $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Are they LI? Yes
- Can they make any vector?

13 2

R^2

1 vector ?

3 vectors ?

u span 2 space, = a basis for 2 space

3 space = 3 LI vectors

eg 3 unit vectors (1 0 0) (0 1 0) (0 0 1)

$\boxed{\text{P64 figure 4.3}}$

Can we form u_1 from u_2 & u ? Why not?

$\therefore$ u_1 u_2 u_3 are LI

can we form all others from u_1 u_2 u_3? Yes.

space = n

vector = pt in nspace

n spa R^n Euclidean Space Why? Hold on

<u>Distance</u> btw 2 pts

$d = d(u, v)$

$u = v$ $d = 0$

$d(u,v) = d(v,u)$

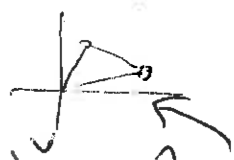
$d \geq 0$

triangular equality

$d(u,v)$ 0 $u = v$

$d(u,v)$ $d(v,u) > 0$ $u \neq v$

$d(u,v)$ $d(u,w) + d(w,v)$ $w \neq u, v$



why?

If v spc
then it is

d may not be unique

Euclidean space has

$u = (a_1, a_2 \dots a_n)$ $v = (b_1 b_2 \dots n)$

$$d(u,v) = \sqrt{(a_1-a_2)^2 + (b_1-b_2)^2 + \dots (a_n\ b_n)^2}$$

L satisfies all properties

Scalar product = 1 x m m 1
= row of
= ' v

$u = (a_1 \dots a_n)$ $v = \dots n)$

$u - v = \ \ .- \ ,\dots a_n - b_n)$

$$= \sqrt{(a_1-b_1)^2 + \dots (a_n - b_n)^2}$$

$$= \sqrt{(u-v)'(u-v)}$$

Commutative, Associative, Distributive

$A + B = B + A$ = comm

$AB \neq BA$
not comm
premultiply
& postmultiply

$(A+B)+C = A + \ \ +C) =$

Associative

$A(BC) = (AB)C$

$m\times n \quad n\times p \quad p\times q$

$m\times n \quad n\times q \qquad m\times p \quad n\times q$

$m\times q \qquad\qquad m\times q$

- $kA = Ak$
- $AB = BA$
  $B = I$
- $AB = BA$
  $B = A^{-1}$

Diagonal mx

square

$$= \begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 0 & a & - \end{bmatrix}$$

$a_{ij} = 0 \quad i \neq j \qquad a_{ij} \neq 0 \quad i = j$

$x' A x$

row mx col

$1\times m \quad m\times m \quad m\times 1$

$= 1\times 1$

if A = diagonal

$x'Ax$ = weighted sum of squares

$\boxed{\text{HW}}$ try with $A = I_n$

$$= \begin{bmatrix} 2 & & \\ & 3 & \\ & & 4 \end{bmatrix}$$

(

egt P69.

Y^o = nat inc

p^o = inf rate

dev Y from Y^o undesirable

$$\Lambda = \alpha(Y - Y^o)^2 + B(p - p^o)^2$$

social loss fn

α, B = weights

why squares?

L pos + neg

L large → v large

Write as diag

$$\begin{bmatrix} Y - Y_0 & p - p_0 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} Y - Y_0 \\ p - p_0 \end{bmatrix}$$

Distributive

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

Identity Mxes & Null

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{I = \text{square}}$ diag

$$\begin{array}{cccc} 3\times3 & 3\times2 & 2\times3 & 3\times3 \\ I & A & = AI & = A \end{array}$$

HW

I here not same dimension

L can insert/delete at random

$A(IB) = (AI)B = AB$

conformability

$I_n^2 = I_n$ & $(I_n)^k = I_n$

check — HW.

$(A)^k = A$ <u>idempotent</u>

<u>Null mx</u> $= O$

$A + O = A = O + A$

$AO = O$

check HW.

✱ $AB = O$

$\Rightarrow A = O$ or $B = O$? No.

eg P72.

✱ $CD = CE$

$\Rightarrow D = E$? No

eg P172.

Transposes & Inverses

$A^T = (A)^T$

$A = A_{m \times n}$ $A^T = A^T_{n \times m}$

$\boxed{\text{swap rows \& cols.}}$

$(A_n)^T = A^T_n$ eg P73

PTO

= 3

− − − =

$(mx$ $x)$

$(mx$ T $= p$

$p \times m$.

1 not

if is square

=

2

3 =

4 -

5 4

- $(A^{-1})^{-1}$

4.1.4 $(AB)^{-1}$ $B^{-}A^{-1}$ (1.1c

- $(A^{T})^{-1}$ $(A$

<u>Proofs</u> P 76

say (AB)

C

P x 1

C

$-$

$$\underset{n\times n}{A}\underset{n\times 1}{x} = \underset{n\times 1}{d}$$

$$A^{-1}Ax = A^{-1}d$$

$$Ix = A^{-1}d$$

$$\underset{n\times 1}{x^*} = \underset{n\times n}{A^{-1}}\underset{n\times 1}{d}$$ unique

Now how to find A^{-1}?
See next chapter.

<u>Check ex 6 P78</u>

$$Y = C + I_0 + G_0$$

$$C = a + bY$$

Solve for Y, C Given $\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}^{-1} = \frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}$

$$Y - C = I_0 + G_0$$

$$-bY + C = a$$

$$\underset{A}{\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}}\underset{x}{\begin{bmatrix} Y \\ C \end{bmatrix}} = \underset{d}{\begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}}$$

PTO

$$x = A^{-1}d \qquad x = \frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}\begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

ite Markov Chains

used to measure/estimate movements
er time, us trans on mx

L elements are prob of moving rom
1 to another

L initial di tn
states

A vector x mx to get changes over
time

Company andes A & B
how many employees will
in A tomorrow.

= no emp in
today

+ no emp
in B

= Pr

B

$x_t' = [A_t \quad B_t]$ = row
= distn emp
over A & B

23

$$M = \begin{bmatrix} P_{AA} & P_{AB} \\ P_{BA} & P_{BB} \end{bmatrix}$$

$$\underset{1\times 2}{x_{t+1}'} = \underset{1\times 2}{x_t'}\, \underset{2\times 2}{M}$$

$$[A_t \quad B_t] \begin{bmatrix} P_{AA} & P_{AB} \\ P_{BA} & P_{BB} \end{bmatrix}$$

$$= \begin{bmatrix} A_t P_{AA} + B_t P_{BA} & A_t P_{AB} + B_t P_{BB} \\ \text{Amt in A} & \text{Amt in B} \end{bmatrix}$$

A_{t+1} $\qquad$ B_{t+1}

=

What is x_{t+2}' ?

$x_{t+2}' = x_{t+1}' M$

$x_{t+2}' = x_t' M M$

$x_{t+2}' = x_t' M^2$

$x_{t+n} = x_t' M^n$

PTO

M = markov transition mx

if n = exog, M = finite
m...

ob

$x =$

$= \;\;] \begin{bmatrix} 0. & \\ 0.4 & 0\,6 \end{bmatrix}$

x_q' x_0'

m

?

sorbing

employees can exit

E person from A

choo to ex

A

P_{EA} P_{EB}

dont

n

$$\left.\right] \begin{bmatrix} P_{AA} & P_{AB} & P_{AE} \\ P_{BA} & P_{BB} & P_{BE} \\ P_{EA} & P_{EE} & P_{EE} \end{bmatrix}^n$$

$$\begin{bmatrix} " & & " \\ & & \\ & & 1 \end{bmatrix}$$

A_s $n \to \infty$ A_n, B_n

$A_0 + B +$